

ALMOST GLOBAL EXISTENCE FOR EXTERIOR NEUMANN PROBLEMS OF SEMILINEAR WAVE EQUATIONS IN 2D

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ABSTRACT. The aim of this article is to prove an “almost” global existence result for some semilinear wave equations in the plane outside a bounded convex obstacle with the Neumann boundary condition.

1. INTRODUCTION

Let \mathcal{O} be an open bounded convex domain with smooth boundary in \mathbb{R}^2 and put $\Omega := \mathbb{R}^2 \setminus \overline{\mathcal{O}}$. Let ∂_ν denote the outer normal derivative on $\partial\Omega$.

We consider the mixed problem for semilinear wave equations in Ω with the Neumann boundary condition:

$$\begin{aligned} (\partial_t^2 - \Delta)u &= G(\partial_t u, \nabla_x u), & (t, x) &\in (0, \infty) \times \Omega, \\ \partial_\nu u(t, x) &= 0, & (t, x) &\in (0, \infty) \times \partial\Omega, \\ u(0, x) &= \phi(x), & x &\in \Omega, \\ \partial_t u(0, x) &= \psi(x), & x &\in \Omega, \end{aligned} \tag{1.1}$$

where ϕ and ψ are \mathcal{C}^∞ -functions compactly supported in $\overline{\Omega}$, and $G : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a nonlinear function. We will study the case of the cubic nonlinearity with small initial data and obtain an estimate from below for the lifespan of the solution in terms of the size of the initial data. Here by the expression “small initial data” we mean that there exist $m \in \mathbb{N}$, $s \in \mathbb{R}$ and a small number $\varepsilon > 0$ such that

$$\|\phi\|_{H^{m+1,s}(\Omega)} + \|\psi\|_{H^{m,s}(\Omega)} \leq \varepsilon,$$

where the weighted Sobolev space $H^{m,s}(\Omega)$ is endowed with the norm

$$\|\varphi\|_{H^{m,s}(\Omega)}^2 := \sum_{|\alpha| \leq m} \int_{\Omega} (1 + |x|^2)^s |\partial_x^\alpha \varphi(x)|^2 dx. \tag{1.2}$$

A large amount of works has been devoted to the study of the mixed problem for nonlinear wave equations in an exterior domain $\Omega \subset \mathbb{R}^n$ for $n \geq 3$, mostly with the Dirichlet boundary condition. To our knowledge very few results deal with the global existence or the lifespan estimate for the exterior mixed problems of nonlinear wave equations in 2D; in [SSW11] the global existence for the case of the Dirichlet boundary condition and the nonlinear terms depending only on u is considered; in [K12] one of the authors obtained an almost global existence result for small initial data under the assumptions that $|G(\partial u)| \simeq (\partial u)^3$, the obstacle is star-shaped and the boundary condition is of the Dirichlet type (see Remark 1.4 below for the detail).

Here we will treat the problem with the Neumann boundary condition in 2D and obtain an analogous result to [K12]. However, because we have a weaker decay property for the solution to

the Neumann exterior problem of linear wave equations in 2D (see Secchi and Shibata [SS03]), we will obtain a slightly worse lifespan estimate than in the Dirichlet case.

For simplicity, we assume that the nonlinear function G in (1.1) is a homogeneous polynomial of cubic order. Equivalently, writing $\partial u = (\partial_t u, \nabla_x u)$, this means that

$$G(\partial u) = \sum_{0 \leq \alpha \leq \beta \leq \gamma \leq 2} g_{\alpha, \beta, \gamma} (\partial_\alpha u) (\partial_\beta u) (\partial_\gamma u) \quad (1.3)$$

with $g_{\alpha, \beta, \gamma} \in \mathbb{R}$ and $(\partial_0, \partial_1, \partial_2) := (\partial_t, \partial_{x_1}, \partial_{x_2})$.

As usual, to consider smooth solutions to the mixed problem, we need some compatibility conditions (see [KK08]). Note that, for a nonnegative integer k and a smooth function $u = u(t, x)$ on $[0, T) \times \Omega$, we have

$$\partial_t^k (G(\partial u)) = G^{(k)}[u, \partial_t u, \dots, \partial_t^{k+1} u], \quad (1.4)$$

where for \mathcal{C}^1 functions $(p_0, p_1, \dots, p_{k+1})$ we put

$$\begin{aligned} G^{(k)}[p_0, p_1, \dots, p_{k+1}] &= \sum_{k_1+k_2+k_3=k} g_{0,0,0} p_{k_1+1} p_{k_2+1} p_{k_3+1} + \sum_{k_1+k_2+k_3=k} \sum_{\gamma=1}^2 g_{0,0,\gamma} p_{k_1+1} p_{k_2+1} (\partial_\gamma p_{k_3}) \\ &+ \sum_{k_1+k_2+k_3=k} \sum_{1 \leq \beta \leq \gamma \leq 2} g_{0,\beta,\gamma} p_{k_1+1} (\partial_\beta p_{k_2}) (\partial_\gamma p_{k_3}) \\ &+ \sum_{k_1+k_2+k_3=k} \sum_{1 \leq \alpha \leq \beta \leq \gamma \leq 2} g_{\alpha,\beta,\gamma} (\partial_\alpha p_{k_1}) (\partial_\beta p_{k_2}) (\partial_\gamma p_{k_3}). \end{aligned}$$

Definition 1.1. *To the mixed problem (1.1) we can associate the recurrence sequence $\{v_j\}_{j \in \mathbb{N}^*}$ with $v_j : \bar{\Omega} \rightarrow \mathbb{R}$ such that*

$$\begin{aligned} v_0 &= \phi, \\ v_1 &= \psi, \\ v_j &= \Delta v_{j-2} + G^{(j-2)}[v_0, v_1, \dots, v_{j-1}], \quad j \geq 2, \end{aligned}$$

where \mathbb{N}^* denotes the set of nonnegative integers and $G^{(k)}$ is defined as above (cf. (1.4)). We say that (ϕ, ψ, G) satisfies the compatibility condition of infinite order in Ω for (1.1) if $\phi, \psi \in \mathcal{C}^\infty(\bar{\Omega})$, and one has

$$\partial_\nu v_j(x) = 0, \quad x \in \partial\Omega$$

for all $j \in \mathbb{N}^*$.

Our aim is to prove the following result.

Theorem 1.1. *Let \mathcal{O} be a convex obstacle. Consider the semilinear mixed problem (1.1) with given compactly supported initial data $(\phi, \psi) \in \mathcal{C}^\infty(\bar{\Omega}) \times \mathcal{C}^\infty(\bar{\Omega})$ and a given nonlinear term $G(\partial u)$ which is a homogeneous polynomial of cubic order as in (1.3). Assume that (ϕ, ψ, G) satisfies the compatibility condition of infinite order in Ω for (1.1).*

Under these assumptions, there exist $\varepsilon_0 > 0$, $m \in \mathbb{N}$, $s \in \mathbb{R}$ such that, if $\varepsilon \in (0, \varepsilon_0]$ and

$$\|\phi\|_{H^{m+1,s}(\Omega)} + \|\psi\|_{H^{m,s}(\Omega)} \leq \varepsilon, \quad (1.5)$$

then the mixed problem (1.1) admits a unique solution $u \in C^\infty([0, T_\varepsilon) \times \Omega)$ with

$$T_\varepsilon \geq \exp(C\varepsilon^{-1}), \quad (1.6)$$

where $C > 0$ is a suitable constant which is uniform with respect to $\varepsilon \in (0, \varepsilon_0]$.

Remark 1.2. The only point where we require that the obstacle \mathcal{O} is convex is to gain the local energy decay (see Lemma 7.5 below). In general one can treat the obstacles for which Lemma 7.5 holds. Unfortunately, for the Neumann problems in 2D, up to our knowledge it is not known if there exists non-convex obstacles satisfying such a local energy decay.

Remark 1.3. One can ask if it is possible to gain a global existence result maintaining our assumption on the growth of G . In general the answer to this question is negative since the blow-up in finite time occurs for $F = (\partial_t u)^3$ when $n = 2$. Indeed, it was proved in [G93] that for any $R > 0$ we can find initial data such that the blow-up for the corresponding Cauchy problem occurs in the region $|x| > t + R$. This result shows the blow-up for the exterior problem with any boundary condition if we choose sufficiently large R , because the solution in $|x| > t + R$ is not affected by the obstacle and the boundary condition, thanks to the finite propagation property (see [KK12] for the corresponding discussion in 3D).

In order to look for global solutions one could investigate the exterior problem with suitable nonlinearity satisfying the so-called *null condition*.

Remark 1.4. If we consider the Cauchy problem in \mathbb{R}^2 , or the Dirichlet problem in a domain exterior to a star-shaped obstacle in 2D, an analogous result to Theorem 1.1 holds with

$$T_\varepsilon \geq \exp(C\varepsilon^{-2}), \quad (1.7)$$

and this lifespan estimate is known to be sharp (see [G93] for the Cauchy problem and [K12] for the Dirichlet problem). One loss of the logarithmic factor in the decay estimates causes this difference between the lifespan estimates (1.6) and (1.7) (see Theorem 2.1 and Remark 7.1 below). It is an interesting problem whether our lower bound (1.6) is sharp or not for the Neumann problem.

2. PRELIMINARIES

In this section we introduce some notation which will be used throughout this paper and some basic lemmas for the proof of Theorem 1.1.

Throughout the paper we shall assume $0 \in \mathcal{O}$ so that we have $|x| \geq c_0$ for $x \in \Omega$ for some positive constant c_0 . We shall also assume that $\overline{\mathcal{O}} \subset B_1$, where B_r stands for an open ball with radius r centered at the origin of \mathbb{R}^2 . Thus a function $v = v(x)$ on Ω vanishing for $|x| \leq 1$ can be naturally regarded as a function on \mathbb{R}^2 .

2.1. Notation. Let us start with some standard notation.

- We put $\langle y \rangle := \sqrt{1 + |y|^2}$ for $y \in \mathbb{R}^d$ with $d \in \mathbb{N}$.

- Let $A = A(y)$ and $B = B(y)$ be two positive functions of some variable y , such as $y = (t, x)$ or $y = x$, on suitable domains. We write $A \lesssim B$ if there exists a positive constant C such that $A(y) \leq CB(y)$ for all y in the intersection of the domains of A and B .
- The $L^2(\Omega)$ norm is denoted by $\|\cdot\|_{L^2_\Omega}$, while the norm $\|\cdot\|_{L^2}$ without any other index stands for $\|\cdot\|_{L^2(\mathbb{R}^2)}$. Similar notation will be used for the L^∞ norms.
- For a time-space depending function u satisfying $u(t, \cdot) \in X$ for $0 \leq t < T$ with a Banach space X , we put $\|u\|_{L_T^\infty X} := \sup_{0 \leq t < T} \|u(t, \cdot)\|_X$. For the brevity of the description, we sometimes use the expression $\|h(s, y)\|_{L_t^\infty L_\Omega^\infty}$ with dummy variables (s, y) for a function h on $[0, t) \times \Omega$, which means $\sup_{0 \leq s < t} \|h(s, \cdot)\|_{L_\Omega^\infty}$.
- For $m \in \mathbb{N}$ and $s \in \mathbb{R}$, by $H^{m,s}(\Omega)$ we denote the weighted Sobolev space with norm defined by (1.2). Moreover $H^m(\Omega)$ and $H^m(\mathbb{R}^2)$ are the standard Sobolev spaces.
- We denote by $\mathcal{C}_0^\infty(\overline{\Omega})$ the set of smooth functions defined on $\overline{\Omega}$ which vanish outside B_R for some $R > 1$.

Let $\nu \in \mathbb{R}$. We put

$$w_\nu(t, x) = \langle x \rangle^{-1/2} \langle t - |x| \rangle^{-\nu} + \langle t + |x| \rangle^{-1/2} \langle t - |x| \rangle^{-1/2}.$$

This weight function w_ν will be used repeatedly in the *a priori estimates of the solution u to (1.1)*. We shall often use the following inequality

$$w_\nu(t, x) \lesssim \langle t + |x| \rangle^{-1/2} (\min\{\langle x \rangle, \langle t - |x| \rangle\})^{-1/2}, \quad \nu \geq 1/2. \quad (2.1)$$

For $\nu, \kappa > 0$ we put

$$W_{\nu, \kappa}(t, x) = \langle t + |x| \rangle^\nu (\min\{\langle x \rangle, \langle t - |x| \rangle\})^\kappa.$$

Finally, for $a \geq 1$ we set

$$\Omega_a = \Omega \cap B_a.$$

Since $\overline{\Omega} \subset B_1$, we see that $\Omega_a \neq \emptyset$ for any $a \geq 1$.

2.2. Vector fields associated with the wave operator. We introduce the vector fields :

$$\Gamma_0 := \partial_0 = \partial_t, \quad \Gamma_1 := \partial_1 = \partial_{x_1}, \quad \Gamma_2 := \partial_2 = \partial_{x_2}, \quad \Gamma_3 := \Lambda := x_1 \partial_2 - x_2 \partial_1.$$

Denoting $[A, B] := AB - BA$, we have

$$[\Gamma_i, \partial_t^2 - \Delta] = 0, \quad i = 0, \dots, 3, \quad (2.2)$$

and also

$$\begin{aligned} [\Gamma_i, \Gamma_j] &= 0, & i, j &= 0, 1, 2, \\ [\Gamma_0, \Gamma_3] &= 0, \\ [\Gamma_1, \Gamma_3] &= \Gamma_2, \\ [\Gamma_2, \Gamma_3] &= -\Gamma_1. \end{aligned}$$

Hence, for $i, j = 0, 1, 2, 3$, we have $[\Gamma_i, \Gamma_j] = \sum_{k=0}^3 c_{ij}^k \Gamma_k$ with suitable constants c_{ij}^k . Moreover, for $i = 0, 1, 2$ and $j = 0, 1, 2, 3$ we also have $[\partial_i, \Gamma_j] = \sum_{k=1}^2 d_{ij}^k \partial_k$ with suitable constants d_{ij}^k .

We put $\partial = (\partial_0, \partial_1, \partial_2)$, $\partial_x = (\partial_1, \partial_2)$, $\Gamma = (\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3) = (\partial, \Lambda)$ and $\tilde{\Gamma} = (\Gamma_1, \Gamma_2, \Gamma_3) = (\partial_x, \Lambda) = (\nabla_x, \Lambda)$. The standard multi-index notation will be used for these sets of vector fields, such as $\partial^\alpha = \partial_0^{\alpha_0} \partial_1^{\alpha_1} \partial_2^{\alpha_2}$ with $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ and $\Gamma^\gamma = \Gamma_0^{\gamma_0} \cdots \Gamma_3^{\gamma_3}$ with $\gamma = (\gamma_0, \dots, \gamma_3)$.

For $\rho \geq 0$, $k \in \mathbb{N}$ and functions $v_0 = v_0(x)$ and $v_1 = v_1(x)$, we put

$$\begin{aligned} \mathcal{A}_{\rho,k}[v_0, v_1] &:= \sum_{|\gamma| \leq k} (\|\langle \cdot \rangle^\rho \tilde{\Gamma}^\gamma v_0\|_{L^\infty_\Omega} + \|\langle \cdot \rangle^\rho \tilde{\Gamma}^\gamma \nabla_x v_0\|_{L^\infty_\Omega} + \|\langle \cdot \rangle^\rho \tilde{\Gamma}^\gamma v_1\|_{L^\infty_\Omega}); \\ \mathcal{B}_{\rho,k}[v_0, v_1] &:= \sum_{|\gamma| \leq k} (\|\langle \cdot \rangle^\rho \tilde{\Gamma}^\gamma v_0\|_{L^\infty} + \|\langle \cdot \rangle^\rho \tilde{\Gamma}^\gamma \nabla_x v_0\|_{L^\infty} + \|\langle \cdot \rangle^\rho \tilde{\Gamma}^\gamma v_1\|_{L^\infty}). \end{aligned}$$

These quantities will be used to control the influence of the initial data to the L^∞ norms of the solution.

Using the vector fields in $\tilde{\Gamma}$, we obtain the following Sobolev-type inequality.

Lemma 2.1. *Let $v \in C_0^2(\overline{\Omega})$. Then we have*

$$\sup_{x \in \Omega} |x|^{1/2} |v(x)| \lesssim \sum_{\substack{|\alpha| + \beta \leq 2 \\ \beta \neq 2}} \|\partial_x^\alpha \Lambda^\beta v\|_{L^2(\Omega)}.$$

Proof. It is well known that for $w \in C_0^2(\mathbb{R}^2)$ we have

$$|x|^{1/2} |w(x)| \lesssim \sum_{\substack{|\alpha| + \beta \leq 2 \\ \beta \neq 2}} \|\partial_x^\alpha \Lambda^\beta w\|_{L^2(\mathbb{R}^2)}, \quad x \in \mathbb{R}^2 \quad (2.3)$$

(see Klainerman [Kl85] for the proof).

Let $\chi = \chi(x)$ be a nonnegative smooth function satisfying $\chi(x) \equiv 0$ for $|x| \leq 1$ and $\chi(x) \equiv 1$ for $|x| \geq 2$. If we rewrite v as $v = \chi v + (1 - \chi)v$, then we have $\chi v \in C_0^\infty(\mathbb{R}^2)$ and (2.3) leads to

$$\sup_{x \in \Omega} |x|^{1/2} |v(x)| \lesssim \sum_{\substack{|\alpha| + \beta \leq 2 \\ \beta \neq 2}} \|\partial_x^\alpha \Lambda^\beta (\chi v)\|_{L^2(\mathbb{R}^2)} + \|(1 - \chi)v\|_{L^\infty(\Omega)}.$$

By using the Sobolev embedding to estimate the last term, we arrive at

$$\sup_{x \in \Omega} |x|^{1/2} |v(x)| \lesssim \sum_{\substack{|\alpha| + \beta \leq 2 \\ \beta \neq 2}} \|\partial_x^\alpha \Lambda^\beta v\|_{L^2(\Omega)} + \sum_{|\alpha| \leq 2} \|\partial_x^\alpha v\|_{L^2(\Omega)}.$$

This completes the proof. \square

2.3. Elliptic estimates. The following elliptic estimates will be used in the energy estimates.

Lemma 2.2. *Let $R > 1$, m be an integer with $m \geq 2$ and $v \in H^m(\Omega)$ such that $\partial_\nu v = 0$ on $\partial\Omega$. Then we have*

$$\|\partial_x^\alpha v\|_{L^2(\Omega)} \lesssim \|\Delta v\|_{H^{|\alpha|-2}(\Omega)} + \|v\|_{H^{|\alpha|-1}(\Omega_{R+1})} \quad (2.4)$$

for $2 \leq |\alpha| \leq m$.

Proof. Let χ be a $C_0^\infty(\mathbb{R}^n)$ function such that $\chi(x) \equiv 1$ for $|x| \leq R$ and $\chi(x) \equiv 0$ for $|x| \geq R+1$. We set $v_1 = \chi v$ and $v_2 = (1 - \chi)v$, so that $v = v_1 + v_2$.

If we put $h = \Delta v_1$, the function v_1 solves the elliptic problem

$$\begin{cases} \Delta v_1 = h & \text{on } \Omega_{R+1}, \\ \partial_\nu v_1 = 0 & \text{on } \partial\Omega, \\ v_1 = 0 & \text{on } \partial B_{R+1}. \end{cases}$$

From Theorem 15.2 of [ADN59], we have

$$\|v_1\|_{H^l(\Omega_{R+1})} \lesssim \|h\|_{H^{l-2}(\Omega_{R+1})} + \|v_1\|_{L^2(\Omega_{R+1})} = \|\Delta v_1\|_{H^{l-2}(\Omega_{R+1})} + \|v_1\|_{L^2(\Omega_{R+1})} \quad (2.5)$$

for $l \geq 2$. Hence

$$\begin{aligned} \|\partial_x^\alpha v_1\|_{L^2(\Omega)} &\lesssim \|\Delta v\|_{H^{|\alpha|-2}(\Omega_{R+1})} + \|\nabla v\|_{H^{|\alpha|-2}(\Omega_{R+1})} + \|v\|_{H^{|\alpha|-2}(\Omega_{R+1})} \\ &\lesssim \|\Delta v\|_{H^{|\alpha|-2}(\Omega_{R+1})} + \|v\|_{H^{|\alpha|-1}(\Omega_{R+1})} \end{aligned}$$

Now we consider v_2 . Note that v_2 can be regarded as a function in \mathbb{R}^2 and we can write $\|\partial_x^\alpha v_2\|_{L^2(\Omega)} = \|\partial_x^\alpha v_2\|_{L^2(\mathbb{R}^2)}$. Let us recall that $\|\partial_x^\beta w\|_{L^2(\mathbb{R}^n)} \lesssim \|\Delta w\|_{L^2(\mathbb{R}^n)}$ for any $w \in H^2(\mathbb{R}^n)$ and $|\beta| = 2$. Writing $\alpha = \beta + \gamma$ with $|\beta| = 2$ and $|\gamma| = |\alpha| - 2$, we have

$$\begin{aligned} \|\partial_x^\alpha v_2\|_{L^2(\Omega)} &\lesssim \|\Delta \partial_x^\gamma v_2\|_{L^2(\mathbb{R}^2)} \lesssim \|\Delta v_2\|_{H^{|\alpha|-2}(\mathbb{R}^2)} \\ &\lesssim \|\Delta v\|_{H^{|\alpha|-2}(\Omega)} + \|v\|_{H^{|\alpha|-1}(\Omega_{R+1})}. \end{aligned}$$

Combining this inequality with the estimate for v_1 , we find (2.4). \square

2.4. Decay estimates for the linear wave equation with Neumann boundary condition.

Given $T > 0$, we consider the mixed problem

$$\begin{aligned} (\partial_t^2 - \Delta)u &= f, & (t, x) &\in (0, T) \times \Omega, \\ \partial_\nu u(t, x) &= 0, & (t, x) &\in (0, T) \times \partial\Omega, \\ u(0, x) &= u_0(x), & x &\in \Omega, \\ (\partial_t u)(0, x) &= u_1(x), & x &\in \Omega. \end{aligned} \quad (2.6)$$

It is known that for $u_0 \in H^2(\Omega)$, $u_1 \in H^1(\Omega)$ and $f \in \mathcal{C}^1([0, T]; L^2(\Omega))$, the mixed problem (2.6) admits a unique solution

$$u \in \bigcap_{j=0}^2 \mathcal{C}^j([0, T]; H^{2-j}(\Omega)),$$

provided that (u_0, u_1, f) satisfies the compatibility condition of order 0, that is to say,

$$\partial_\nu u_0(x) = 0, \quad x \in \partial\Omega \quad (2.7)$$

(see [I68] for instance). Under these assumptions for $\vec{u}_0 := (u_0, u_1)$, the solution u of (2.6) will be denoted by $S[\vec{u}_0, f](t, x)$. We set $K[\vec{u}_0](t, x)$ for the solution of (2.6) with $f \equiv 0$ and $L[f](t, x)$ for the solution of (2.6) with $\vec{u}_0 \equiv (0, 0)$; in other words we put

$$K[\vec{u}_0](t, x) := S[\vec{u}_0, 0](t, x), \quad L[f](t, x) := S[(0, 0), f](t, x)$$

so that we get

$$S[\vec{u}_0, f](t, x) = K[\vec{u}_0](t, x) + L[f](t, x),$$

where $K[\vec{u}_0]$ and $L[f]$ are well defined because both of $(u_0, u_1, 0)$ and $(0, 0, f)$ satisfy the compatibility condition of order 0. In order to obtain a smooth solution to (2.6), we need the compatibility condition of infinite order.

Definition 2.1. Suppose that u_0, u_1 and f are smooth. Define u_j for $j \geq 2$ inductively by

$$u_j(x) = \Delta u_{j-2}(x) + (\partial_t^{j-2} f)(0, x), \quad j \geq 2.$$

We say that (u_0, u_1, f) satisfies the compatibility condition of infinite order in Ω for (2.6), if one has

$$\partial_\nu u_j = 0 \quad \text{on} \quad \partial\Omega$$

for any nonnegative integer j .

We say that $(u_0, u_1, f) \in X(T)$ if the following three conditions are satisfied:

- $(u_0, u_1) \in \mathcal{C}_0^\infty(\overline{\Omega}) \times \mathcal{C}_0^\infty(\overline{\Omega})$,
- $f \in C^\infty([0, T] \times \overline{\Omega})$; moreover, $f(t, \cdot) \in \mathcal{C}_0^\infty(\overline{\Omega})$ for any $t \in [0, T]$,
- (u_0, u_1, f) satisfies the compatibility condition of infinite order.

It is known that if $(u_0, u_1, f) \in X(T)$, then we have $S[\vec{u}_0, f] \in \mathcal{C}^\infty([0, T] \times \overline{\Omega})$ (see [I68] for instance).

The following decay estimates play important roles in our proof of the main theorem.

Theorem 2.1. Let \mathcal{O} be a convex set and k be a nonnegative integer. Suppose that $\Xi = (\vec{u}_0, f) = (u_0, u_1, f) \in X(T)$.

(i) Let $\mu > 0$. Then we have

$$\sum_{|\delta| \leq k} |\Gamma^\delta S[\Xi](t, x)| \lesssim \mathcal{A}_{2+\mu, 3+k}[\vec{u}_0] + \log(e+t) \sum_{|\delta| \leq 3+k} \| |y|^{1/2} W_{1,1+\mu}(s, y) \Gamma^\delta f(s, y) \|_{L_t^\infty L_\Omega^\infty} \quad (2.8)$$

for $(t, x) \in [0, T] \times \overline{\Omega}$.

(ii) Let $0 < \eta < 1/2$ and $\mu > 0$. Then we have

$$\begin{aligned} w_{(1/2)-\eta}^{-1}(t, x) \sum_{|\delta| \leq k} |\Gamma^\delta \partial S[\Xi](t, x)| &\lesssim \\ &\lesssim \mathcal{A}_{2+\mu, k+4}[\vec{u}_0] + \log^2(e+t+|x|) \sum_{|\delta| \leq k+4} \| |y|^{1/2} W_{1,1}(s, y) \Gamma^\delta f(s, y) \|_{L_t^\infty L_\Omega^\infty}, \end{aligned} \quad (2.9)$$

$$\begin{aligned} w_{1/2}^{-1}(t, x) \sum_{|\delta| \leq k} |\Gamma^\delta \partial S[\Xi](t, x)| &\lesssim \\ &\lesssim \mathcal{A}_{2+\mu, k+4}[\vec{u}_0] + \log^2(e+t+|x|) \sum_{|\delta| \leq k+4} \| |y|^{1/2} W_{1,1+\mu}(s, y) \Gamma^\delta f(s, y) \|_{L_t^\infty L_\Omega^\infty} \end{aligned} \quad (2.10)$$

for $(t, x) \in [0, T] \times \overline{\Omega}$.

(iii) Let $0 < \eta < 1$ and $\mu > 0$. Then we have

$$\begin{aligned} w_{1-\eta}^{-1}(t, x) \sum_{|\delta| \leq k} |\Gamma^\delta \partial \partial_t S[\Xi](t, x)| &\lesssim \\ &\lesssim \mathcal{A}_{2+\mu, k+5}[\vec{u}_0] + \log^2(e + t + |x|) \sum_{|\delta| \leq k+5} \| |y|^{1/2} W_{1,1}(s, y) \Gamma^\delta f(s, y) \|_{L_t^\infty L_\Omega^\infty} \end{aligned} \quad (2.11)$$

for $(t, x) \in [0, T) \times \overline{\Omega}$.

We will prove Theorem 2.1 in Section 7 below, by using the so-called cut-off method to combine the corresponding decay estimates for the Cauchy problem with the local energy decay.

3. THE ABSTRACT ARGUMENT FOR THE PROOF OF THE MAIN THEOREM

Since the local existence of smooth solutions for the mixed problem (1.1) has been shown by [SN89] (see also the Appendix), what we need to do for showing the large time existence of the solution is to derive suitable *a priori* estimates: following [SN89], we need the control of $\|u(t)\|_{H^9(\Omega)} + \|\partial_t u(t)\|_{H^8(\Omega)}$ for the solution u .

Let u be the local solution of (1.1), assuming (1.5) holds for large $m \in \mathbb{N}$ and $s > 0$. Let T^* be the supremum of T such that (1.1) admits a (unique) classical solution in $[0, T) \times \overline{\Omega}$. For $0 < T \leq T^*$, a small $\eta > 0$, and nonnegative integers H and K we define

$$\begin{aligned} \mathcal{E}_{H,K}(T) &\equiv \sum_{|\gamma| \leq H-1} \|w_{1/2}^{-1} \Gamma^\gamma \partial u\|_{L_T^\infty L_\Omega^\infty} + \sum_{1 \leq j+|\alpha| \leq K} \|\partial_t^j \partial_x^\alpha u\|_{L_T^\infty L_\Omega^2} \\ &+ \sum_{|\delta| \leq K-2} \|\langle s \rangle^{-1/2} \Gamma^\delta \partial u(s, y)\|_{L_T^\infty L_\Omega^2} + \sum_{|\delta| \leq K-8} \|\langle s \rangle^{-(1/4)-\eta} \Gamma^\delta \partial u(s, y)\|_{L_T^\infty L_\Omega^2} \\ &+ \sum_{|\delta| \leq K-14} \|\langle s \rangle^{-2\eta} \Gamma^\delta \partial u(s, y)\|_{L_T^\infty L_\Omega^2} + \sum_{|\delta| \leq K-20} \|\Gamma^\delta \partial u\|_{L_T^\infty L_\Omega^2}. \end{aligned}$$

We neglect the first sum when $H = 0$. Similarly we neglect summations taken over the empty set as K varies. We also put

$$\mathcal{E}_{H,K}(0) = \lim_{T \rightarrow 0^+} \mathcal{E}_{H,K}(T).$$

Observe that $\mathcal{E}_{H,K}(0)$ can be determined only by ϕ , ψ and G and that we have

$$\mathcal{E}_{H,K}(0) \lesssim \|\phi\|_{H^{m+1,s}(\Omega)} + \|\psi\|_{H^{m,s}(\Omega)}$$

for suitably large $m \in \mathbb{N}$ and $s > 0$ depending on H and K . From (1.5) for such $m \in \mathbb{N}$ and $s > 0$, we see that $\mathcal{E}_{H,K}(0)$ is finite. The previous inequality can be obtained combining the embedding $H^r(\Omega) \hookrightarrow L^\infty(\Omega)$ for $r > 1$ with the trivial inequality $|\Gamma_3 f| \leq \langle x \rangle |\partial_1 f| + \langle x \rangle |\partial_2 f|$ and the equivalence between $\sum_{|\alpha| \leq m} \|\langle \cdot \rangle^s \partial_x^\alpha f\|_{L_\Omega^2}$ and $\|\langle \cdot \rangle^s f\|_{H^m(\Omega)}$. In order to optimize m or s it is possible to use sharpest embedding theorem in weighted Sobolev spaces proved for example in [GL04].

Our goal is to show the following claim.

Claim 3.1. We can take suitable H and K and sufficiently large m and s , so that there exist positive numbers C_1 , P and Q and a strictly increasing continuous function $\mathcal{R} : [0, \infty) \rightarrow [0, \infty)$ with $\mathcal{R}(0) = 0$ such that if $\mathcal{E}_{H,K}(T) \leq 1$, then

$$\mathcal{E}_{H,K}(T) \leq C_1 \varepsilon + \mathcal{R}(\mathcal{E}_{H,K}^P(T) \log^Q(e + T)) (\varepsilon + \mathcal{E}_{H,K}(T)), \quad (3.1)$$

provided that (1.5) holds with $\varepsilon \leq 1$. Here C_1 , P , Q and \mathcal{R} are independent of ε and T .

Let us explain how from (3.1) we can gain the lifespan estimate. Suppose that the above claim is true. If we assume (1.5) for some m and s which are sufficiently large, then, as we have mentioned, there exists $C_* > 0$ such that $\mathcal{E}_{H,K}(0) < 2C_*\varepsilon$. We may assume $C_* \geq \max\{C_1, 1\}$. We set $\varepsilon_0 = \min\{(2C_*)^{-1}, 1\}$ and suppose that $0 < \varepsilon \leq \varepsilon_0$, so that we have $\varepsilon \leq 1$ and $2C_*\varepsilon \leq 1$. We put

$$T_*(\varepsilon) := \sup \{T \in [0, T^*) : \mathcal{E}_{H,K}(T) \leq 2C_*\varepsilon\}.$$

In particular, for any $T \leq T_*(\varepsilon)$, we have $\mathcal{E}_{H,K}(T) \leq 1$. From (3.1) with $T = T_*(\varepsilon)$, we get

$$\mathcal{E}_{H,K}(T_*(\varepsilon)) \leq C_*\varepsilon + \mathcal{R}((2C_*\varepsilon)^P \log^Q(e + T_*(\varepsilon))) (3C_*\varepsilon).$$

We are going to prove

$$\mathcal{R}((2C_*\varepsilon)^P \log^Q(e + T^*)) > \frac{1}{4} \quad (3.2)$$

by contradiction. Suppose that T^* satisfies

$$\mathcal{R}((2C_*\varepsilon)^P \log^Q(e + T^*)) \leq \frac{1}{4}. \quad (3.3)$$

Since $T_*(\varepsilon) \leq T^*$, and \mathcal{R} is an increasing function, we obtain

$$\mathcal{E}_{H,K}(T_*(\varepsilon)) \leq \frac{7}{4}C_*\varepsilon < 2C_*\varepsilon.$$

Therefore we get $T_*(\varepsilon) = T^*$, because otherwise the continuity of $\mathcal{E}_{H,K}(T)$ implies that there exists $\tilde{T} > T_*(\varepsilon)$ satisfying $\mathcal{E}_{H,K}(\tilde{T}) \leq 2C_*\varepsilon$, which contradicts the definition of $T_*(\varepsilon)$. However, if $T_*(\varepsilon) = T^*$, and H, K are sufficiently large, we can prove

$$\begin{aligned} \|u\|_{L_{T^*}^\infty H^9(\Omega)} + \|\partial_t u\|_{L_{T^*}^\infty H^8(\Omega)} &\lesssim \varepsilon + (1 + T^*)\mathcal{E}_{H,K}(T^*) \\ &= \varepsilon + (1 + T_*(\varepsilon))\mathcal{E}_{H,K}(T_*(\varepsilon)) \lesssim \varepsilon + (1 + T_*(\varepsilon))2C_*\varepsilon, \end{aligned} \quad (3.4)$$

and we can extend the solution beyond the time T^* by the local existence theorem, which contradicts the definition of T^* . Therefore (3.3) is not true, and we obtain (3.2). This means that, for any $\varepsilon \leq \varepsilon_0$, there exists $\tilde{C} > 0$ such that

$$T^* > \exp\{\tilde{C}\varepsilon^{-P/Q}\}. \quad (3.5)$$

It remains to show (3.4). It is evident that

$$\|u\|_{L_{T^*}^\infty H^9(\Omega)} + \|\partial_t u\|_{L_{T^*}^\infty H^8(\Omega)} \lesssim \|u\|_{L_{T^*}^\infty L_\Omega^2} + \mathcal{E}_{0,9}(T^*).$$

In order to estimate $\|u\|_{L_{T^*}^\infty L_\Omega^2}$ we will use the expression

$$u(t, x) = u(0, x) + \int_0^t \partial_t u(\tau, x) d\tau,$$

which leads to

$$\|u\|_{L_T^\infty L_\Omega^2} \lesssim \varepsilon + T^* \mathcal{E}_{0,1}(T^*).$$

As a conclusion, we obtain (1.6), once we can show that Claim 3.1 is true with $P = Q = 1$. This will be done in the next three sections.

4. ENERGY ESTIMATES FOR THE STANDARD DERIVATIVES

In this section we are going to estimate $\|\partial_t^j \partial_x^\alpha u\|_{L_T^\infty L_\Omega^2}$ for $j + |\alpha| \geq 1$. In the first subsection, we consider the case where $j \geq 0$ and $|\alpha| = 1$. This can be done directly through the standard energy inequalities. In the second subsection, the case where $j \geq 1$ and $|\alpha| \geq 2$ will be treated with the help of the elliptic estimate, Lemma 2.2. In the third subsection, we consider the case where $j = 0$ and $|\alpha| \geq 2$. Lemma 2.2 will be used again, but this time we need the estimate of $\|u\|_{L_T^\infty L^2(\Omega_{R+1})}$ for some $R > 0$, which is not included in the definition of $\mathcal{E}_{H,K}(T)$. Since we are considering the 2D Neumann problem, it seems difficult to use some embedding theorem to estimate $\|u\|_{L_T^\infty L^2(\Omega_{R+1})}$ by $\|\nabla_x u\|_{L_T^\infty H^k(\Omega)}$ with some positive integer k . Instead, we will employ the L^∞ estimate, Theorem 2.1, for this purpose.

4.1. On the energy estimates for the derivatives in time. First we set

$$E(v; t) = \frac{1}{2} \int_\Omega \{|\partial_t v(t, x)|^2 + |\nabla_x v(t, x)|^2\} dx$$

for a smooth function $v = v(t, x)$.

Let j be a nonnegative integer. Since ∂_t commutes with the restriction of the function to $\partial\Omega$, we have $\partial_\nu \partial_t^j u(t, x) = 0$ for all $(t, x) \in (0, T) \times \partial\Omega$. Therefore, by the standard energy method, we find

$$\frac{d}{dt} E(\partial_t^j u; t) = \int_\Omega \partial_t^j (G(\partial u))(t, x) \partial_t^{j+1} u(t, x) dx.$$

Recalling the definition of $\mathcal{E}_{H,K}(T)$, for $j + |\alpha| \geq 1$ we have

$$|\partial_t^j \nabla_x^\alpha u(t, x)| \leq w_{1/2}(t, x) \mathcal{E}_{j+|\alpha|,0}(T), \quad x \in \Omega, t \in [0, T]. \quad (4.1)$$

Applying (4.1) and the Leibniz rule we find

$$\frac{d}{dt} E(\partial_t^j u; t) \lesssim \|w_{1/2}(t)\|_{L_\Omega^\infty}^2 \mathcal{E}_{[j/2]+1,0}^2(T) \sum_{h=0}^j \int_\Omega |\partial_t^h \partial u(t, x)| |\partial_t^{j+1} u(t, x)| dx.$$

It is also clear that if $j + |\alpha| \geq 1$, one has

$$\|\partial_t^j \partial_x^\alpha u(t)\|_{L_\Omega^2} \leq \mathcal{E}_{0,j+|\alpha|}(T), \quad t \in [0, T].$$

This gives

$$\frac{d}{dt} E(\partial_t^j u; t) \lesssim \|w_{1/2}(t)\|_{L_\Omega^\infty}^2 \mathcal{E}_{[j/2]+1,0}^2(T) \mathcal{E}_{0,j+1}^2(T).$$

Since $\mathcal{E}_{H,K}(T)$ is increasing in H and K , we get

$$\frac{d}{dt} E(\partial_t^j u; t) \lesssim \|w_{1/2}(t)\|_{L_\Omega^\infty}^2 \mathcal{E}_{[j/2]+1,j+1}^4(T).$$

As a trivial consequence of (2.1), we find $w_{1/2}(t, x) \leq \langle t \rangle^{-1/2}$, so that

$$\frac{d}{dt} E(\partial_t^j u; t) \lesssim \langle t \rangle^{-1} \mathcal{E}_{[j/2]+1, j+1}^4(T).$$

After integration this gives

$$\sum_{l=0}^j \|\partial_t^{l+1} u(t)\|_{L_\Omega^2} + \sum_{l=0}^j \|\partial_t^l \nabla_x u(t)\|_{L_\Omega^2} \lesssim \mathcal{E}_{j+1}(0) + \mathcal{E}_{j+1}^2(T) \log^{1/2}(e+t) \quad (4.2)$$

for any $j \geq 0$ and $t \in [0, T]$, where

$$\mathcal{E}_s(T) = \mathcal{E}_{[(s-1)/2]+1, s}(T)$$

for any integer $s \geq 0$.

4.2. On the energy estimates for the space-time derivatives. Since the spatial derivatives do not preserve the Neumann boundary condition, we need to use elliptic regularity results.

We shall show that for $j \geq 1$ and $k \geq 0$ it holds

$$\sum_{|\alpha|=k} \|\partial_t^j \partial_x^\alpha u(t)\|_{L_\Omega^2} \lesssim \mathcal{E}_{j+k}(0) + \mathcal{E}_{j+k}^2(T) \log^{1/2}(e+T) + \mathcal{E}_{j+k-1}^3(T) \quad (4.3)$$

with $\mathcal{E}_s(T) = \mathcal{E}_{[(s-1)/2]+1, s}(T)$ as before.

It is clear that (4.3) follows from (4.2) when $j \geq 1$ and $k = 0, 1$.

Next we suppose that (4.3) holds for $j \geq 1$ and $k \leq l$ with some positive integer l . Let $|\alpha| = l+1$ and $j \geq 1$. Since $|\alpha| \geq 2$, we apply to $\partial_t^j u$ the elliptic estimate (Lemma 2.2) and we obtain

$$\|\partial_x^\alpha \partial_t^j u(t)\| \lesssim \|\Delta \partial_t^j u(t)\|_{H^{l-1}(\Omega)} + \|\partial_t^j u(t)\|_{H^l(\Omega)}.$$

By (4.3) for $k \leq l$, we see that the second term has the desired bound. On the other hand, using the fact that u is a solution to (1.1), for the first term we have

$$\|\Delta \partial_t^j u(t)\|_{H^{l-1}(\Omega)} \lesssim \|\partial_t^{j+2} u(t)\|_{H^{l-1}(\Omega)} + \|\partial_t^j (G(\partial u))(t)\|_{H^{l-1}(\Omega)}.$$

Since $(j+2) + (l-1) = j+l+1$, it follows from (4.3) for $k = l-1$ with j replaced by $j+2$ that

$$\|\partial_t^{j+2} u(t)\|_{H^{l-1}(\Omega)} \lesssim \mathcal{E}_{j+l+1}(0) + \mathcal{E}_{j+l+1}^2(T) \log^{1/2}(e+T) + \mathcal{E}_{j+l}^3(T),$$

which is the desired bound. Finally, observing that $w_{1/2}(t, x) \leq 1$, we get

$$\|\partial_t^j G(\partial u)(t)\|_{H^{l-1}(\Omega)} \lesssim \sum_{1 \leq |\beta| \leq [(j+l-1)/2]+1} \|\partial^\beta u(t)\|_{L_\Omega^\infty}^2 \sum_{1 \leq |\gamma| \leq j+l} \|\partial^\gamma u(t)\|_{L_\Omega^2} \lesssim \mathcal{E}_{j+l}^3(T).$$

Combining these estimates, we obtain (4.3) for $j \geq 1$ and $k = l+1$. This completes the proof of (4.3) for $j \geq 1$ and $k \geq 0$.

4.3. On the energy estimates for the space derivatives. Our aim here is to estimate $\|\partial_x^\alpha u\|_{L_T^\infty L_\Omega^2}$ for $|\alpha| = k \geq 1$. The estimate for $k = 1$ is included in (4.2). Let us consider the case $|\alpha| = k \geq 2$. Let us fix $R > 1$. The elliptic estimate (2.4) gives

$$\begin{aligned} \sum_{|\alpha|=k} \|\partial_x^\alpha u\|_{L_T^\infty L_\Omega^2} &\lesssim \|\Delta u\|_{L_T^\infty H^{k-2}(\Omega)} + \|u\|_{L_T^\infty H^{k-1}(\Omega_{R+1})} \\ &\lesssim \|\partial_t^2 u\|_{L_T^\infty H^{k-2}(\Omega)} + \|G(\partial u)\|_{L_T^\infty H^{k-2}(\Omega)} + \|u\|_{L_T^\infty H^{k-1}(\Omega_{R+1})}. \end{aligned}$$

The first term can be estimated by (4.3) and we get

$$\|\partial_t^2 u\|_{L_T^\infty H^{k-2}(\Omega)} \lesssim \mathcal{E}_k(0) + \mathcal{E}_k^2(T) \log^{1/2}(e+T) + \mathcal{E}_{k-1}^3(T).$$

For the second term, we obtain the following inequality as before:

$$\|G(\partial u)\|_{L_T^\infty H^{k-2}(\Omega)} \lesssim \mathcal{E}_{k-1}^3(T).$$

As for the third term, we get

$$\begin{aligned} \|u\|_{L_T^\infty H^{k-1}(\Omega_{R+1})} &\lesssim \sum_{1 \leq |\beta| \leq k-1} \|\partial_x^\beta u\|_{L_T^\infty L^2(\Omega_{R+1})} + \|u\|_{L_T^\infty L^2(\Omega_{R+1})} \\ &\lesssim \sum_{1 \leq |\beta| \leq k-1} \|\partial_x^\beta u\|_{L_T^\infty L_\Omega^2} + \|u\|_{L_T^\infty L^\infty(\Omega_{R+1})}. \end{aligned}$$

Now we fix $\mu \in (0, 1/2)$ and use (2.8) with $k = 0$ to obtain

$$\|u\|_{L_T^\infty L_\Omega^\infty} \lesssim \mathcal{A}_{2+\mu,3}[\phi, \psi] + \log(e+T) \sum_{|\delta| \leq 3} \left\| \langle y \rangle^{1/2} W_{1,1+\mu}(s, y) \Gamma^\delta(G(\partial u))(s, y) \right\|_{L_T^\infty L_\Omega^\infty}. \quad (4.4)$$

By using (2.1), for any $s \in [0, T)$ we have

$$\sum_{|\delta| \leq 3} |\Gamma^\delta G(\partial u)(s, y)| \lesssim \langle s + |y| \rangle^{-3/2} (\min\{\langle y \rangle, \langle |y| - s \rangle\})^{-3/2} \mathcal{E}_{4,0}^3(T).$$

This implies

$$\sum_{|\delta| \leq 3} \left\| |y|^{1/2} W_{1,1+\mu}(s, y) \Gamma^\delta(G(\partial u))(s, y) \right\|_{L_T^\infty L_\Omega^\infty} \lesssim \mathcal{E}_{4,0}^3(T),$$

and (4.4) gives

$$\|u\|_{L_T^\infty L_\Omega^\infty} \lesssim \mathcal{A}_{2+\mu,3}[\phi, \psi] + \mathcal{E}_{4,0}^3(T) \log(e+T). \quad (4.5)$$

Summing up the estimates above, for $|\alpha| = k \geq 2$, we get

$$\begin{aligned} \sum_{|\alpha|=k} \|\partial_x^\alpha u\|_{L_T^\infty L_\Omega^2} &\leq \mathcal{A}_{2+\mu,3}[\phi, \psi] + \mathcal{E}_k(0) + \mathcal{E}_k^2(T) \log^{1/2}(e+T) + \mathcal{E}_{k-1}^3(T) + \mathcal{E}_{4,0}^3 \log(e+T) \\ &\quad + \sum_{1 \leq |\alpha| \leq k-1} \|\partial_x^\alpha u\|_{L_T^\infty L_\Omega^2}. \end{aligned}$$

Finally we inductively obtain

$$\sum_{|\alpha|=k} \|\partial_x^\alpha u\|_{L_T^\infty L_\Omega^2} \leq \mathcal{A}_{2+\mu,3}[\phi, \psi] + \mathcal{E}_k(0) + \mathcal{E}_k^2(T) \log^{1/2}(e+T) + \mathcal{E}_{k-1}^3(T) + \mathcal{E}_{4,0}^3(T) \log(e+T)$$

for $k \geq 1$.

4.4. Conclusion for the energy estimates of the standard derivatives. If m and s are sufficiently large, (1.5) and the Sobolev embedding theorem lead to

$$\mathcal{A}_{2+\mu,3}[\phi, \psi] + \mathcal{E}_K(0) \lesssim \|\phi\|_{H^{m+1,s}(\Omega)} + \|\psi\|_{H^{m,s}(\Omega)} \lesssim \varepsilon.$$

Summing up the estimates in this section, we get

$$\sum_{1 \leq j+|\alpha| \leq K} \|\partial_t^j \partial_x^\alpha u\|_{L_T^\infty L_\Omega^2} \lesssim \varepsilon + \mathcal{E}_K^2(T) \log^{1/2}(e+T) + \mathcal{E}_K^3(T) \log(e+T) \quad (4.6)$$

for each $K \geq 7$.

5. ON THE ENERGY ESTIMATES FOR THE GENERALIZED DERIVATIVES

Throughout this section and the next one, we suppose that K is sufficiently large, and we assume that $\mathcal{E}_K(T) \leq 1$.

5.1. Direct energy estimates for the generalized derivatives. Let $|\delta| \leq K-2$. Recalling (2.2), it follows that

$$\begin{aligned} \frac{d}{dt} E(\Gamma^\delta u; t) &= \int_{\Omega} \Gamma^\delta G(\partial u)(t, x) \partial_t \Gamma^\delta u(t, x) dx \\ &\quad + \int_{\partial\Omega} \nu \cdot \nabla_x \Gamma^\delta u(t, x) \partial_t \Gamma^\delta u(t, x) dS =: I_\delta(t) + II_\delta(t), \end{aligned} \quad (5.1)$$

where $\nu = \nu(x)$ is the unit outer normal vector at $x \in \partial\Omega$ and dS is the surface measure on $\partial\Omega$.

Since $G(\partial u)$ is a homogeneous polynomial of order three, we can say that

$$|\Gamma^\delta G(\partial u) \partial_t \Gamma^\delta u| \lesssim \sum_{|\delta_1| \leq \lfloor |\delta|/2 \rfloor} |\Gamma^{\delta_1} \partial u|^2 \sum_{|\delta_2| \leq |\delta|} |\Gamma^{\delta_2} \partial u(t, x)|^2. \quad (5.2)$$

Applying the Hölder inequality and taking the L^∞ norm of the first factor, we arrive at

$$|I_\delta(t)| \lesssim \langle t \rangle^{-1} \mathcal{E}_{\lfloor |\delta|/2 \rfloor + 1, 0}^2(T) \langle t \rangle \mathcal{E}_{0, K}^2(T) \lesssim \mathcal{E}_K^4(T), \quad (5.3)$$

since $|\delta| \leq K-2$.

Now we treat the boundary term, by means of the trace theorem. Since $\partial\Omega \subset B_1$, the norms of the generalized derivatives on $\partial\Omega$ are equivalent to the norms of the standard derivatives. Hence for all $t \in (0, T)$ we have

$$|II_\delta(t)| \lesssim \sum_{1 \leq |\gamma| + k \leq |\delta| + 1} \|\partial_t^k \partial_x^\gamma u(t)\|_{L^2(\partial\Omega)}^2.$$

Moreover, by the trace theorem and (4.6), we see that

$$|II_\delta(t)| \lesssim \sum_{1 \leq |\gamma| + k \leq |\delta| + 2} \|\partial_t^k \partial_x^\gamma u(t)\|_{L_\Omega^2}^2 \lesssim \left(\varepsilon + \mathcal{R}_0(\mathcal{E}_K(T) \log^{1/2}(e+T)) \mathcal{E}_K(T) \right)^2,$$

because of the assumption $|\delta| \leq K-2$. Here we put

$$\mathcal{R}_0(s) = s + s^2.$$

Summarizing the above estimates, for any $K \geq 7$ and $|\delta| \leq K - 2$, it holds

$$\begin{aligned} \frac{d}{dt} E(\Gamma^\delta u; t) &\lesssim \left(\varepsilon + \mathcal{R}_0(\mathcal{E}_K(T) \log^{1/2}(e + T)) \mathcal{E}_K(T) \right)^2 + \mathcal{E}_K^4(T) \\ &\lesssim \left(\varepsilon + \mathcal{R}_0(\mathcal{E}_K(T) \log^{1/2}(e + T)) \mathcal{E}_K(T) \right)^2. \end{aligned}$$

For the last inequality, we recall that $\mathcal{E}_K(T) \leq 1$. After integration, this gives

$$\begin{aligned} \sum_{|\delta| \leq K-2} \|\Gamma^\delta \partial u(t)\|_{L_\Omega^2} &\lesssim \mathcal{E}_K(0) + t^{1/2} \left(\varepsilon + \mathcal{R}_0(\mathcal{E}_K(T) \log^{1/2}(e + T)) \mathcal{E}_K(T) \right) \\ &\lesssim \langle t \rangle^{1/2} \left(\varepsilon + \mathcal{R}_0(\mathcal{E}_K(T) \log^{1/2}(e + T)) \mathcal{E}_K(T) \right). \end{aligned} \quad (5.4)$$

5.2. Refinement of the energy estimates for the generalized derivatives. Let $1 \leq |\delta| \leq K - 8$. Since $\partial\Omega$ is a bounded set, it follows from (5.1) that

$$\begin{aligned} |H_\delta(t)| &\lesssim \|\Gamma^\delta \partial_t u(t)\|_{L^2(\partial\Omega)} \sum_{|\gamma| \leq |\delta|} \|\Gamma^\gamma \nabla_x u(t)\|_{L^2(\partial\Omega)} \\ &\lesssim \sum_{1 \leq |\gamma| \leq \delta} \|\partial^\gamma \partial_t u(t)\|_{L^\infty(\partial\Omega)} \sum_{|\gamma| \leq |\delta|} \|\partial^\gamma \nabla_x u(t)\|_{L^\infty(\partial\Omega)}. \end{aligned}$$

Since we have $|x| \leq 1$ for $x \in \partial\Omega$, we get $\langle |x| + t \rangle \simeq \langle t \rangle \simeq \langle |x| - t \rangle$ for $x \in \partial\Omega$. In particular we get $\sup_{x \in \partial\Omega} w_\nu(t, x) \lesssim \langle t \rangle^{-\nu}$ for $0 < \nu \leq 1$. We fix sufficiently small and positive constants $0 < \eta < 1/4$ and $\mu > 0$. Applying the pointwise estimates (2.9) and (2.11) in Theorem 2.1, we get

$$|H_\delta(t)| \lesssim \langle t \rangle^{-(3/2)+\eta} \log^4(e + t) \left(\mathcal{A}_{2+\mu, |\delta|+4}^2[\phi, \psi] + A_{|\delta|+4}^2(t) \right),$$

where

$$A_s(t) = \sum_{|\gamma| \leq s} \left\| |y|^{1/2} W_{1,1}(s, y) \Gamma^\gamma (G(\partial u))(s, y) \right\|_{L_t^\infty L_\Omega^\infty}.$$

If m and s are sufficiently large, by the Sobolev embedding theorem we have $A_{2+\mu, |\delta|+4}[\phi, \psi] \lesssim \varepsilon$ and we obtain

$$|H_\delta(t)| \lesssim \langle t \rangle^{-(3/2)+\eta} \log^4(e + t) \left(\varepsilon^2 + A_{|\delta|+4}^2(t) \right). \quad (5.5)$$

In order to estimate $A_{|\delta|+4}(t)$, we argue as in (5.2), so that

$$\sum_{|\gamma| \leq |\delta|+4} |\Gamma^\gamma G(\partial u)(s, y)| \lesssim w_{1/2}^2(s, y) \mathcal{E}_{[(|\delta|+4)/2]+1,0}^2(T) \sum_{|\gamma'| \leq |\delta|+4} |\Gamma^{\gamma'} \partial u(s, y)|.$$

Now using (2.1) and applying Lemma 2.1 to estimate $|\Gamma^{\gamma'} \partial u|$, we obtain

$$\sum_{|\gamma| \leq |\delta|+4} |\Gamma^\gamma G(\partial u)(s, y)| \lesssim |y|^{-1/2} W_{1,1}^{-1}(s, y) \mathcal{E}_{[(|\delta|+4)/2]+1,0}^2(T) \sum_{|\gamma| \leq |\delta|+6} \|\Gamma^\gamma \partial u(s, \cdot)\|_{L_\Omega^2},$$

which yields

$$A_{|\delta|+4}(t) \lesssim \mathcal{E}_K^2(T) \sum_{|\gamma| \leq |\delta|+6} \|\Gamma^\gamma \partial u(s, y)\|_{L_t^\infty L_\Omega^2} \quad (5.6)$$

because we have $[(|\delta| + 4)/2] \leq [(K - 1)/2]$ for $|\delta| \leq K - 8$. Observing that

$$\sum_{|\gamma| \leq |\delta|+6} \|\Gamma^\gamma \partial u(s, y)\|_{L_t^\infty L_\Omega^2} \lesssim \langle t \rangle^{1/2} \mathcal{E}_K(T)$$

for $|\delta| \leq K - 8$, we see from (5.5) and (5.6) that

$$|II_\delta(t)| \lesssim \langle t \rangle^{-(1/2)+2\eta} (\varepsilon^2 + \mathcal{E}_K^6(T)).$$

Moreover for $|\delta| \leq K - 8$ the inequality (5.3) can be improved as

$$|I_\delta(t)| \lesssim \langle t \rangle^{-1} \mathcal{E}_{[|\delta|/2]+1,0}^2(T) \left(\langle t \rangle^{1/4+\eta} \mathcal{E}_{0,K}(T) \right)^2 \lesssim \langle t \rangle^{-(1/2)+2\eta} \mathcal{E}_K^4(T).$$

Coming back to (5.1), one can conclude from the assumption $\mathcal{E}_K(T) \leq 1$ that

$$\begin{aligned} \sum_{1 \leq |\delta| \leq K-8} \|\Gamma^\delta \partial u(t)\|_{L_\Omega^2} &\lesssim \mathcal{E}_K(0) + \langle t \rangle^{(1/4)+\eta} (\varepsilon + \mathcal{E}_K^2(T)) \\ &\lesssim \langle t \rangle^{(1/4)+\eta} (\varepsilon + \mathcal{E}_K^2(T)). \end{aligned} \quad (5.7)$$

Next step is to improve this estimate for lower $|\delta|$ in order to avoid the polynomial growth in t . Let $1 \leq |\delta| \leq K - 14$. From (5.6) and the definition of $\mathcal{E}_K(T)$ we get

$$A_{|\delta|+4}(t) \lesssim \mathcal{E}_K^3(T) \langle t \rangle^{(1/4)+\eta}.$$

From (5.5), it follows that

$$\begin{aligned} |II_\delta(t)| &\lesssim \langle t \rangle^{-(3/2)+\eta} \log^4(e+t) \left(\varepsilon^2 + \langle t \rangle^{(1/2)+2\eta} \mathcal{E}_K^6(T) \right) \\ &\lesssim \langle t \rangle^{-1+4\eta} (\varepsilon^2 + \mathcal{E}_K^6(T)). \end{aligned}$$

On the other hand, for $|\delta| \leq K - 14$ it holds

$$|I_\delta(t)| \lesssim \langle t \rangle^{-1} \mathcal{E}_{[|\delta|/2]+1,0}^2(T) \left(\langle t \rangle^{2\eta} \mathcal{E}_{0,K}(T) \right)^2 \lesssim \langle t \rangle^{-1+4\eta} \mathcal{E}_K^4(T).$$

Summing up these estimates and integrating (5.1), we get

$$\sum_{1 \leq |\delta| \leq K-14} \|\Gamma^\delta \partial u(t)\|_{L_\Omega^2} \lesssim \langle t \rangle^{2\eta} (\varepsilon + \mathcal{E}_K^2(T)). \quad (5.8)$$

We repeat the above procedure once again with $1 \leq |\delta| \leq K - 20$. Being $|\delta| + 6 \leq K - 14$, from (5.6) we have $A_{|\delta|+4}(t) \lesssim \langle t \rangle^{2\eta} \mathcal{E}_K^3(T)$. In turn this implies

$$\begin{aligned} |II_\delta(t)| &\lesssim \langle t \rangle^{-(3/2)+\eta} \log^4(e+t) (\varepsilon^2 + \langle t \rangle^{4\eta} \mathcal{E}_K^6(T)) \\ &\lesssim \langle t \rangle^{-(3/2)+6\eta} (\varepsilon^2 + \mathcal{E}_K^6(T)). \end{aligned}$$

In this case $I_\delta(t) \leq \langle t \rangle^{-1} \mathcal{E}_K^4(T)$. After integration we get

$$\begin{aligned} \sum_{1 \leq |\delta| \leq K-20} \|\Gamma^\delta \partial u(t)\|_{L_\Omega^2} &\lesssim \mathcal{E}_K(0) + \mathcal{E}_K^2(T) \log^{1/2}(e+t) + \varepsilon + \mathcal{E}_K^3(T) \\ &\lesssim \varepsilon + \mathcal{E}_K^2(T) \log^{1/2}(e+t). \end{aligned} \quad (5.9)$$

This estimate is the best we can obtain with our methods due to the estimate of $I_\delta(t)$.

6. BOUNDEDNESS FOR THE L^∞ NORM AND THE CONCLUSION OF THE PROOF OF THEOREM 1.1

Summarizing (4.6), (5.4), (5.7), (5.8), (5.9) we have

$$\mathcal{E}_{0,K}(T) \lesssim \varepsilon + \mathcal{R}_0(\mathcal{E}_{[(K-1)/2]+1,K}(T) \log^{1/2}(e+T)) \mathcal{E}_{[(K-1)/2]+1,K}(T) \quad (6.1)$$

with $K \geq 20$ and $\mathcal{R}_0(s) = s + s^2$. If $\mathcal{E}_{H,0}(T)$ with $H = [(K-1)/2] + 1$ has the same bound of $\mathcal{E}_{0,K}(T)$ given in (6.1), then we conclude that the estimate (3.1) in the Claim 3.1 holds for $P = 1$ and $Q = 1/2$, and hence $T^* \geq \exp(\tilde{C}\epsilon^{-2})$. However, \mathcal{R}_0 (and hence Q) will be changed due to the following argument. Such a modification yields a worse estimate for the lifespan.

Since we assume $\phi, \psi \in \mathcal{C}_0^\infty(\overline{\Omega})$, there is a positive constant M such that $|x| \leq t + M$ in $\text{supp } u(t, \cdot)$ for $t \geq 0$. Hence we have $\log(e + t + |x|) \lesssim \log(e + t)$ in $\text{supp } u(t, \cdot)$.

From (5.6) and the definition of $\mathcal{E}_K(T)$, it follows that $A_{|\delta|+4}(t) \lesssim \mathcal{E}_K^3(T)$ for $K \geq 26$ and $|\delta| \leq K - 26$. Let $\mu > 0$. Then we have $\mathcal{A}_{2+\mu,K-22}[\phi, \psi] \lesssim \varepsilon$ if m and s are sufficiently large. For fixed $0 < \eta < 1/2$, by (2.9), we obtain

$$\sum_{|\gamma| \leq K-26} |\Gamma^\gamma \partial u(t, x)| \lesssim \mathcal{B}(\varepsilon, t) w_{(1/2)-\eta}(t, x)$$

where

$$\mathcal{B}(\varepsilon, t) := \varepsilon + \log^2(e + t) \mathcal{E}_K^3(T).$$

Using this estimate, we obtain

$$\sum_{|\gamma| \leq K-26} |\Gamma^\gamma G(\partial u)(t, x)| \lesssim w_{1/2}^2(t, x) \mathcal{E}_{[(K-1)/2]+1,0}^2(T) w_{(1/2)-\eta}(t, x) \mathcal{B}(\varepsilon, t).$$

Since $|y|^{1/2} w_{1/2-\eta} \lesssim 1$, this implies

$$\mathcal{A}_{|\delta|+4}(t) \lesssim \mathcal{E}_K^2(T) \mathcal{B}(\varepsilon, t)$$

for any $|\delta| + 4 \leq K - 26$. Therefore, (2.10) in Theorem 2.1 yields

$$\sum_{|\gamma| \leq K-30} |\Gamma^\gamma \partial u(t, x)| \lesssim (\varepsilon + \mathcal{B}(\varepsilon, t) \mathcal{E}_K^2(T) \log^2(e + t)) w_{1/2}(t, x).$$

For $K \geq 61$ we have $[(K-1)/2] + 1 \leq K - 30$, and we conclude that

$$\sum_{|\gamma| \leq [(K-1)/2]+1} \|w_{1/2}^{-1} \Gamma^\gamma \partial u\|_{L_T^\infty L_\Omega^\infty} \lesssim \varepsilon + \mathcal{B}(\varepsilon, t) \mathcal{E}_K^2(T) \log^2(e + T). \quad (6.2)$$

Finally, we combine (6.1) and (6.2) to obtain

$$\begin{aligned} \mathcal{E}_K(T) &\lesssim \varepsilon + (\varepsilon + \mathcal{E}_K(T)) \times \\ &\quad \times \left(\mathcal{E}_K(T) \log^{1/2}(e + T) + \mathcal{E}_K^2(T) \log^2(e + T) + \mathcal{E}_K^4(T) \log^4(e + T) \right). \end{aligned}$$

In order to find

$$\mathcal{E}_K(T) \leq C_1 \varepsilon + \mathcal{R}(\mathcal{E}_K^P(T) \log^Q(e + T)) (\varepsilon + \mathcal{E}_K(T))$$

with as larger P/Q as possible, we take

$$\mathcal{R}(\tau) := C_2(\tau + \tau^2 + \tau^4)$$

and $P = Q = 1$. Recalling the discussion in Section 3, we obtain Theorem 1.1.

7. PROOF OF POINTWISE ESTIMATES

In this section, we go back to the Neumann problem (2.6) and will prove Theorem 2.1 by combining the decay estimates for the Cauchy problem in \mathbb{R}^2 and the local energy decay estimate through the cut-off argument.

7.1. Decomposition of solutions. Recall the definitions of $X(T)$ and $S[\vec{u}_0, f](t, x)$, $K[\vec{u}_0](t, x)$, $L[f](t, x)$ in Subsection 2.4. In the same manner, the solution of the Cauchy problem

$$\begin{aligned} (\partial_t^2 - \Delta)v &= g & (t, x) &\in (0, T) \times \mathbb{R}^2, \\ v(0, x) &= v_0(x), & x &\in \mathbb{R}^2, \\ (\partial_t v)(0, x) &= v_1(x), & x &\in \mathbb{R}^2, \end{aligned} \quad (7.1)$$

will be denoted by $S_0[\vec{v}_0, g](t, x)$ with $\vec{v}_0 = (v_0, v_1)$. Then we have

$$S_0[\vec{v}_0, g](t, x) = K_0[\vec{v}_0](t, x) + L_0[g](t, x),$$

where $K_0[\vec{v}_0](t, x)$ and $L_0[g](t, x)$ are the solutions of (7.1) with $g = 0$ and $\vec{v}_0 = (0, 0)$, respectively. In other words, $K_0[\vec{v}_0](t, x) = S_0[\vec{v}_0, 0](t, x)$ and $L_0[g](t, x) = S_0[(0, 0), g](t, x)$.

Now we proceed to introduce the cut-off argument. For $a > 0$, we denote by ψ_a a smooth radially symmetric function on \mathbb{R}^2 satisfying

$$\begin{cases} \psi_a(x) = 0, & |x| \leq a, \\ \psi_a(x) = 1, & |x| \geq a + 1. \end{cases} \quad (7.2)$$

Lemma 7.1. Fix $a \geq 1$. Let $(u_0, u_1, f) \in X(T)$. Assume that for any $t \in (0, T)$ one has

$$\text{supp } f(t, \cdot) \subset \overline{\Omega_{t+a}} \quad \text{and} \quad \text{supp } u_0 \subset \overline{\Omega_a}, \quad \text{supp } u_1 \subset \overline{\Omega_a}.$$

Then we have

$$S[\vec{u}_0, f](t, x) = \psi_a(x)S_0[\psi_{2a}\vec{u}_0, \psi_{2a}f](t, x) + \sum_{i=1}^4 S_i[\vec{u}_0, f](t, x), \quad (7.3)$$

where

$$S_1[\vec{u}_0, f](t, x) = (1 - \psi_{2a}(x))L[\psi_a, -\Delta]S_0[\psi_{2a}\vec{u}_0, \psi_{2a}f](t, x), \quad (7.4)$$

$$S_2[\vec{u}_0, f](t, x) = -L_0[\psi_{2a}, -\Delta]L[\psi_a, -\Delta]S_0[\psi_{2a}\vec{u}_0, \psi_{2a}f](t, x), \quad (7.5)$$

$$S_3[\vec{u}_0, f](t, x) = (1 - \psi_{3a}(x))S[(1 - \psi_{2a})\vec{u}_0, (1 - \psi_{2a})f](t, x), \quad (7.6)$$

$$S_4[\vec{u}_0, f](t, x) = -L_0[\psi_{3a}, -\Delta]S[(1 - \psi_{2a})\vec{u}_0, (1 - \psi_{2a})f](t, x). \quad (7.7)$$

For the proof, we refer to [K07].

Observe that the first term on the right-hands side of (7.3) can be evaluated by applying the decay estimates for the whole space case. In contrast, the local energy decay estimates for the mixed problem work well in estimating $S_j[\vec{u}_0, f]$ for $1 \leq j \leq 4$, because we always have some localized factor in front of the operators L , S and in their arguments.

7.2. Known estimates for the 2D linear Cauchy problem. In this subsection we recall the decay estimates for solutions of homogeneous wave equation. Since $\Lambda K_0[v_0, v_1] = K_0[\Lambda v_0, \Lambda v_1]$ by (2.2), we find that Proposition 2.1 of [Ku93] leads to the following.

Lemma 7.2. *Let $m \in \mathbb{N}$. For any $(v_0, v_1) \in \mathcal{C}_0^\infty(\mathbb{R}^2) \times \mathcal{C}_0^\infty(\mathbb{R}^2)$, it holds that*

$$\langle t + |x| \rangle^{1/2} \log^{-1} \left(e + \frac{\langle t + |x| \rangle}{\langle t - |x| \rangle} \right) \sum_{|\beta| \leq m} |\Gamma^\beta K_0[v_0, v_1](t, x)| \lesssim \mathcal{B}_{3/2, m}[v_0, v_1]. \quad (7.8)$$

Under the same assumption, for any $\mu > 0$ we have

$$\langle t + |x| \rangle^{1/2} \langle t - |x| \rangle^{1/2} \sum_{|\beta| \leq m} |\Gamma^\beta K_0[v_0, v_1](t, x)| \lesssim \mathcal{B}_{2+\mu, m}[v_0, v_1]. \quad (7.9)$$

For $\kappa \geq 1$ and $\tau \geq 0$, we define

$$\Psi_\kappa(\tau) := \begin{cases} 1, & \kappa > 1, \\ \log(e + \tau), & \kappa = 1. \end{cases}$$

The following two lemmas are proved for $m = 0$ in [D03]. For the general case, see [K12].

Lemma 7.3. *Let $\kappa \geq 1$ and $m \in \mathbb{N}$. Then we have*

$$\sum_{|\delta| \leq m} |\Gamma^\delta L_0[g](t, x)| \lesssim \Psi_\kappa(t + |x|) \sum_{|\delta| \leq m} \|\langle y \rangle^{1/2} W_{1/2, \kappa}(s, y) \Gamma^\delta g(s, y)\|_{L_t^\infty L^\infty}, \quad (7.10)$$

and

$$\begin{aligned} & \langle t + |x| \rangle^{1/2} \log^{-1} \left(e + \frac{\langle t + |x| \rangle}{\langle t - |x| \rangle} \right) \sum_{|\delta| \leq m} |\Gamma^\delta L_0[g](t, x)| \lesssim \\ & \lesssim \Psi_\kappa(t + |x|) \sum_{|\delta| \leq m} \|\langle y \rangle^{1/2} W_{1, \kappa}(s, y) \Gamma^\delta g(s, y)\|_{L_t^\infty L^\infty} \end{aligned} \quad (7.11)$$

for any $(t, x) \in [0, T) \times \mathbb{R}^2$.

Lemma 7.4. *Let $0 < \sigma < 3/2$, $\kappa > 1$, $\mu \geq 0$, $0 < \eta < 1$ and $m \in \mathbb{N}$. Then, for any $(t, x) \in [0, T) \times \mathbb{R}^2$, one has*

$$\begin{aligned} & \sum_{|\delta| \leq m} |\Gamma^\delta \partial L_0[g](t, x)| \lesssim \\ & \lesssim w_\sigma(t, x) \Psi_{\mu+1}(t + |x|) \sum_{|\delta| \leq m+1} \|\langle y \rangle^{1/2+\kappa} \langle s + |y| \rangle^{\sigma+\mu} \Gamma^\delta g(s, y)\|_{L_t^\infty L^\infty}, \end{aligned} \quad (7.12)$$

$$\begin{aligned} & \sum_{|\delta| \leq m} |\Gamma^\delta \partial L_0[g](t, x)| \lesssim \\ & \lesssim w_{1-\eta}(t, x) \log(e + t + |x|) \sum_{|\delta| \leq m+1} \|\langle y \rangle^{1/2} W_{1,1}(s, y) \Gamma^\delta g(s, y)\|_{L_t^\infty L^\infty}. \end{aligned} \quad (7.13)$$

7.3. The local energy decay estimates. We come back to the linear problem (2.6). Let $X_a(T)$ be the set of all $(u_0, u_1, f) \in X(T)$ such that

$$u_0(x) = u_1(x) = 0 \text{ for } |x| \geq a, \quad (7.14)$$

$$f(t, x) = 0 \text{ for } |x| \geq a, \ t \in [0, T]. \quad (7.15)$$

The following local energy decay will be used in the proof of the pointwise estimate.

Lemma 7.5. *Assume that \mathcal{O} is convex. Let $a, b > 1$, $\gamma \in (0, 1]$ and $m \in \mathbb{N}$. If $\Xi = (u_0, u_1, f) \in X_a(T)$, then for any $t \in [0, T)$ one has*

$$\begin{aligned} \sum_{|\alpha| \leq m} \langle t \rangle^\gamma \|\partial^\alpha S[\Xi](t)\|_{L^2(\Omega_b)} &\lesssim \\ &\lesssim \|u_0\|_{H^m(\Omega)} + \|u_1\|_{H^{m-1}(\Omega)} + \log(e+t) \sum_{|\alpha| \leq m-1} \|\langle s \rangle^\gamma (\partial^\alpha f)(s, y)\|_{L_t^\infty L_\Omega^2}. \end{aligned} \quad (7.16)$$

Proof. For $a, b > 1$, it is known that there exists a positive constant $C = C(a, b)$ such that

$$\begin{aligned} \int_{\Omega_b} (|\partial_t K[\vec{\phi}_0](t, x)|^2 + |\nabla_x K[\vec{\phi}_0](t, x)|^2 + |K[\vec{\phi}_0](t, x)|^2) dx &\lesssim \\ &\lesssim \langle t \rangle^{-2} \left(\|\phi_0\|_{H^1(\Omega)}^2 + \|\phi_1\|_{L^2(\Omega)}^2 \right) \end{aligned} \quad (7.17)$$

for any $\vec{\phi}_0 = (\phi_0, \phi_1) \in H^2(\Omega) \times H^1(\Omega)$ satisfying $\phi_0(x) = \phi_1(x) \equiv 0$ for $|x| \geq a$ and satisfying also the compatibility condition of order 0, that is to say, $\partial_\nu \phi_0(x) = 0$ for $x \in \partial\Omega$ (see for instance Lemma 2.1 of [SS03]; see also Morawetz [M75] and Vainberg [V75]).

Now let $(u_0, u_1, f) \in X_a(T)$ with some $a > 1$. Let u_j for $j \geq 2$ be defined as in Definition 2.1. Then, by Duhamel's principle, it follows that

$$\begin{aligned} \partial_t^j S[(u_0, u_1, f)](t, x) &= \\ &= K[(u_j, u_{j+1})](t, x) + \int_0^t K[(0, (\partial_t^j f)(s))](t-s, x) ds \end{aligned} \quad (7.18)$$

for any nonnegative integer $j \in \mathbb{N}^*$ and any $(t, x) \in [0, T) \times \Omega$. Observe that $(u_j, u_{j+1}, 0)$ satisfies the compatibility condition of order 0, because $(u_0, u_1, f) \in X(T)$ implies $\partial_\nu u_j = 0$ on $\partial\Omega$; the compatibility condition of order 0 is also trivially satisfied for $(0, (\partial_s^j f)(s), 0)$ for all $s \geq 0$.

Therefore, by (7.17) we have

$$\begin{aligned} \sum_{|\alpha| \leq 1} \|\partial^\alpha K[u_j, u_{j+1}](t)\|_{L^2(\Omega_b)} &\lesssim \langle t \rangle^{-1} (\|u_j\|_{H^1(\Omega)} + \|u_{j+1}\|_{L^2(\Omega)}) \\ &\lesssim \langle t \rangle^{-1} (\|u_0\|_{H^{j+1}(\Omega)} + \|u_1\|_{H^j(\Omega)} + \sum_{k=0}^{j-1} \|(\partial_t^k f)(0)\|_{L^2(\Omega)}) \end{aligned}$$

and

$$\begin{aligned} \sum_{|\alpha| \leq 1} \int_0^t \|\partial^\alpha K[(0, (\partial_t^j f)(s))](t-s)\|_{L^2(\Omega_b)} ds &\lesssim \int_0^t \langle t-s \rangle^{-1} \|(\partial_t^j f)(s)\|_{L^2(\Omega)} ds \\ &\lesssim \langle t \rangle^{-\gamma} \log(e+t) \sup_{0 \leq s \leq t} \langle s \rangle^\gamma \|(\partial_t^j f)(s)\|_{L^2(\Omega)} \end{aligned}$$

for any $\gamma \in (0, 1]$. In conclusion for any $j \in \mathbb{N}^*$, we have

$$\begin{aligned} \sum_{|\alpha| \leq 1} \|\partial^\alpha \partial_t^j S[(u_0, u_1, f)](t)\|_{L^2(\Omega_b)} &\lesssim \\ &\lesssim \langle t \rangle^{-\gamma} (\|u_0\|_{H^{j+1}(\Omega)} + \|u_1\|_{H^j(\Omega)} + \sum_{k=0}^j \log(e+t) \sup_{0 \leq s \leq t} \langle s \rangle^\gamma \|(\partial_t^k f)(s)\|_{L^2(\Omega)}). \end{aligned} \quad (7.19)$$

In order to evaluate $\partial^\alpha S[\Xi]$ for $2 \leq |\alpha| \leq m$, we have only to combine (7.19) with a variant of (2.4):

$$\|\varphi\|_{H^m(\Omega_b)} \lesssim \|\Delta_x \varphi\|_{H^{m-2}(\Omega_{b'})} + \|\varphi\|_{H^{m-1}(\Omega_{b'})}, \quad (7.20)$$

where $1 < b < b'$ and $\varphi \in H^m(\Omega)$ with $m \geq 2$; we can easily obtain (7.20) from (2.4) by cutting off φ for $|x| \geq b'$.

In order to complete the proof, one has to apply this inequality recalling the equation $\Delta S[\Xi] = \partial_t^2 S[\Xi] - f$. Invoking (7.19), we finally get the basic estimate (7.17). \square

7.4. Proof of Theorem 2.1. The following lemma is the main tool for the proof of Theorem 2.1.

Lemma 7.6. *Let \mathcal{O} be a convex set. Let $a, b > 1$, $0 < \rho \leq 1$, $m \in \mathbb{N}^*$ and $\kappa \geq 1$.*

(i) *Suppose that χ is a smooth function on \mathbb{R}^2 satisfying $\text{supp } \chi \subset B_b$. If $\Xi = (u_0, u_1, f) \in X_a(T)$, then*

$$\begin{aligned} \langle t \rangle^\rho \sum_{|\delta| \leq m} |\Gamma^\delta(\chi S[\Xi])(t, x)| &\lesssim \\ &\lesssim \|u_0\|_{H^{m+2}(\Omega)} + \|u_1\|_{H^{m+1}(\Omega)} + \log(e+t) \sum_{|\beta| \leq m+1} \|\langle s \rangle^\rho \partial^\beta f(s, y)\|_{L_t^\infty L^\infty(\Omega_a)} \end{aligned} \quad (7.21)$$

for $(t, x) \in [0, T) \times \overline{\Omega}$.

(ii) *Let $g \in \mathcal{C}^\infty([0, T) \times \mathbb{R}^2)$ such that $\text{supp } g(t, \cdot) \subset \overline{B_a} \setminus \overline{B_1}$ for any $t \in [0, T)$. Then*

$$\sum_{|\delta| \leq m} |\Gamma^\delta L_0[g](t, x)| \lesssim \sum_{|\beta| \leq m} \|\langle s \rangle^{1/2} \partial^\beta g(s, y)\|_{L_t^\infty L^\infty(\Omega_a)}, \quad (7.22)$$

and for any $0 \leq \eta < \rho$ we have

$$w_{\rho-\eta}^{-1}(t, x) \sum_{|\delta| \leq m} |\Gamma^\delta \partial L_0[g](t, x)| \lesssim \Psi_{\eta+1}(t + |x|) \sum_{|\beta| \leq m+1} \|\langle s \rangle^\rho \partial^\beta g(s, y)\|_{L_t^\infty L^\infty(\Omega_a)}. \quad (7.23)$$

for $(t, x) \in [0, T) \times \overline{\Omega}$.

(iii) *Let $(v_0, v_1, g) \in \mathcal{C}^\infty(\mathbb{R}^2) \times \mathcal{C}^\infty(\mathbb{R}^2) \times \mathcal{C}^\infty([0, T) \times \mathbb{R}^2)$. If $v_0 = v_1 = g(t, \cdot) = 0$ for any $x \in B_1$ and $t \in [0, T)$, then*

$$\begin{aligned} \langle t \rangle^{1/2} \sum_{|\beta| \leq m} |\Gamma^\beta S_0[v_0, v_1, g](t, x)| &\lesssim \\ &\lesssim \mathcal{A}_{3/2, m}[v_0, v_1] + \Psi_\kappa(t + |x|) \sum_{|\beta| \leq m} \|\langle y \rangle^{1/2} W_{1, \kappa}(s, y) \Gamma^\beta g(s, y)\|_{L_t^\infty L^\infty(\Omega)} \end{aligned} \quad (7.24)$$

for $(t, x) \in [0, T) \times \overline{\Omega_b}$.

Proof. First we note that for any smooth function $h : [0, T) \times \overline{\Omega} \rightarrow \mathbb{R}$ such that $\text{supp } h(t, \cdot) \subset B_R$ for any $t \in [0, T)$ and suitable $R > 1$, it holds that

$$\sum_{|\beta| \leq m} |\Gamma^\beta h(t, x)| \lesssim \sum_{|\beta| \leq m} |\partial^\beta h(t, x)|. \quad (7.25)$$

Clearly the same estimate holds for $h : [0, T) \times \mathbb{R}^2 \rightarrow \mathbb{R}$.

We start with the proof of (7.21). Let $\Xi \in X_a(T)$ and $0 < \rho \leq 1$. For $(t, x) \in [0, T) \times \overline{\Omega}$, combining (7.25) with the standard Sobolev inequality and then applying the local energy decay (7.16), we get

$$\begin{aligned} \langle t \rangle^\rho \sum_{|\beta| \leq m} |\Gamma^\beta (\chi S[\Xi])(t, x)| &\lesssim \langle t \rangle^\rho \sum_{|\beta| \leq m+2} \|\partial^\beta S[\Xi](t)\|_{L^2(\Omega_b)} \\ &\lesssim \|u_0\|_{H^{m+2}(\Omega)} + \|u_1\|_{H^{m+1}(\Omega)} + \log(e+t) \sum_{|\beta| \leq m+1} \|\langle s \rangle^\rho \partial^\beta f(s, y)\|_{L_t^\infty L_\Omega^2}. \end{aligned}$$

Since $\text{supp } f(t, \cdot) \subset \overline{\Omega}_a$ implies $\|\partial^\beta f(s)\|_{L^2(\Omega)} \lesssim \|\partial^\beta f(s)\|_{L^\infty(\Omega_a)}$, we obtain (7.21).

Next we prove (7.22) by the aid of the decay estimates for the linear Cauchy problem. By (7.10) for some $\kappa > 1$, we find

$$\sum_{|\delta| \leq m} |\Gamma^\delta L_0[g](t, x)| \lesssim \sum_{|\delta| \leq m} \|\langle y \rangle^{1/2} W_{1/2, \kappa}(s, y) \Gamma^\delta g(s, y)\|_{L_t^\infty L^\infty}.$$

Using the assumption $\text{supp } g(t, \cdot) \subset \overline{B_a \setminus B_1} \subset \overline{\Omega}_a$, we gain (7.22).

Similarly, if we use (7.12) (with σ being replaced by $\rho - \eta$ and μ by η), instead of (7.10), then we get (7.23).

Finally we prove (7.24) by using (7.8) and (7.11). It follows that

$$\begin{aligned} \langle t + |x| \rangle^{1/2} \log \left(e + \frac{\langle t + |x| \rangle}{\langle t - |x| \rangle} \right) \sum_{|\beta| \leq m} |\Gamma^\beta S_0[\vec{v}_0, g](t, x)| &\lesssim \\ &\lesssim \mathcal{B}_{3/2, m}[\vec{v}_0] + \Psi_\kappa(t + |x|) \sum_{|\beta| \leq m} \|\langle y \rangle^{1/2} W_{1, \kappa}(s, y) \Gamma^\beta g(s, y)\|_{L_t^\infty L^\infty} \end{aligned}$$

for $(t, x) \in [0, T) \times \mathbb{R}^2$. Observe that the logarithmic term on the left-hand side is equivalent to a constant when $x \in \overline{\Omega}_b$. Thus we get (7.24), because our assumption ensures that support of data and $\text{supp } g(t, \cdot)$ are contained in Ω . This completes the proof. \square

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. According to Lemma 7.1 with $a = 1$, we can write

$$S[\Xi](t, x) = \psi_1(x) S_0[\psi_2 \Xi](t, x) + \sum_{i=1}^4 S_i[\Xi](t, x) \quad (7.26)$$

for $(t, x) \in [0, T) \times \overline{\Omega}$, where ψ_a is defined by (7.2) and $S_i[\Xi]$ for $1 \leq i \leq 4$ are defined by (7.4)–(7.7) with $a = 1$. It is easy to check that

$$[\psi_a, -\Delta]h(t, x) = h(t, x) \Delta \psi_a(x) + 2 \nabla_x h(t, x) \cdot \nabla_x \psi_a(x) \quad (7.27)$$

for $(t, x) \in [0, T) \times \overline{\Omega}$, $a \geq 1$ and any smooth function h . Note that this identity implies

$$(0, 0, [\psi_a, -\Delta]h) \in X_{a+1}(T) \quad (7.28)$$

because $\text{supp } \nabla_x \psi_a \cup \text{supp } \Delta \psi_a \subset \overline{B_{a+1}} \setminus \overline{B_a}$.

First we prove (2.8). Applying (7.8) and (7.11), we have

$$\begin{aligned} \langle t + |x| \rangle^{1/2} \log^{-1} \left(e + \frac{\langle t + |x| \rangle}{\langle t - |x| \rangle} \right) \sum_{|\delta| \leq k} \left| \Gamma^\delta S_0[\psi_2 \Xi](t, x) \right| &\lesssim \\ &\lesssim \mathcal{B}_{3/2, k}[\psi_2 \vec{u}_0] + \sum_{|\delta| \leq k} \| \langle y \rangle^{1/2} W_{1, 1+\mu}(s, y) \Gamma^\delta(\psi_2 f)(s, y) \|_{L_t^\infty L^\infty} \\ &\lesssim \mathcal{A}_{3/2, k}[\vec{u}_0] + \sum_{|\delta| \leq k} \| |y|^{1/2} W_{1, 1+\mu}(s, y) \Gamma^\delta f(s, y) \|_{L_t^\infty L_\Omega^\infty}, \end{aligned}$$

so that

$$\begin{aligned} \langle t + |x| \rangle^{1/2} \log^{-1} \left(e + \frac{\langle t + |x| \rangle}{\langle t - |x| \rangle} \right) \sum_{|\delta| \leq k} \left| \Gamma^\delta(\psi_1(x) S_0[\psi_2 \Xi](t, x)) \right| &\lesssim \\ &\lesssim \mathcal{A}_{3/2, k}[\vec{u}_0] + \sum_{|\delta| \leq k} \| |y|^{1/2} W_{1, 1+\mu}(s, y) \Gamma^\delta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned} \quad (7.29)$$

Now we write

$$S_1[\Xi] = (1 - \psi_2) L[[\psi_1, -\Delta] K_0[\psi_2 \vec{u}_0]] + (1 - \psi_2) L[[\psi_1, -\Delta] L_0[\psi_2 f]] =: S_{1,1}[\Xi] + S_{1,2}[\Xi].$$

We can apply (7.21) to estimate $S_{1,2}[\Xi]$, because we have $L[h] = S[0, 0, h]$ and $\text{supp}(1 - \psi_2) \subset B_3$ and because (7.28) guarantees $(0, 0, [\psi_1, -\Delta] L_0[\psi_2 f]) \in X_2$. Therefore we get

$$\begin{aligned} \langle t \rangle^{1/2} \sum_{|\delta| \leq k} |\Gamma^\delta S_{1,2}[\Xi](t, x)| &\lesssim \log(e + t) \sum_{|\beta| \leq k+1} \| \langle s \rangle^{1/2} \partial^\beta ([\psi_1, -\Delta] L_0[\psi_2 f])(s, x) \|_{L_t^\infty L^\infty(\Omega_2)} \\ &\lesssim \log(e + t) \sum_{|\beta| \leq k+2} \| \langle s \rangle^{1/2} \partial^\beta L_0[\psi_2 f](s, x) \|_{L_t^\infty L^\infty(\Omega_2)}, \end{aligned}$$

where we have used (7.27) to obtain the second line. Recalling that $L_0[h] = S_0[0, 0, h]$ and noting that $\psi_2 f(t, x) = 0$ if $|x| \leq 2$, we can use (7.24) to obtain

$$\langle t \rangle^{1/2} \sum_{|\delta| \leq k} |\Gamma^\delta S_{1,2}[\Xi](t, x)| \lesssim \log(e + t) \sum_{|\beta| \leq k+2} \| |y|^{1/2} W_{1, 1+\mu}(s, y) \Gamma^\beta f(s, y) \|_{L_t^\infty L_\Omega^\infty} \quad (7.30)$$

for $(t, x) \in [0, T) \times \overline{\Omega}$.

In order to estimate $S_{1,1}[\Xi]$, we combine the Sobolev embedding and the local energy decay estimate (7.16) with $\gamma = 1$. Then we get

$$\begin{aligned} \sum_{|\delta| \leq k} |\Gamma^\delta S_{1,1}[\Xi](t, x)| &\lesssim \|(1 - \psi_2)L[[\psi_1, -\Delta]K_0[\psi_2 \vec{u}_0]](t, \cdot)\|_{H^{2+k}(\Omega)} \\ &\lesssim \|S[0, 0, [\psi_1, -\Delta]K_0[\psi_2 \vec{u}_0]](t, \cdot)\|_{H^{2+k}(\Omega_3)} \\ &\lesssim \langle t \rangle^{-1} \log(e+t) \sum_{|\delta| \leq k+1} \|\langle s \rangle \partial^\delta ([\psi_1, -\Delta]K_0[\psi_2 \vec{u}_0])(s, y)\|_{L_t^\infty L_\Omega^2} \\ &\lesssim \langle t \rangle^{-1} \log(e+t) \sum_{|\beta| \leq k+2} \|\langle s \rangle \partial^\beta K_0[\psi_2 \vec{u}_0](s, y)\|_{L_t^\infty L^\infty(\Omega_2)}. \end{aligned}$$

Then we use (7.9); recalling that we are in a bounded y -domain, for any $\mu > 0$ we get

$$\langle t \rangle^{1/2} \langle t + |x| \rangle^{1/2} \log^{-1}(e+t) \sum_{|\delta| \leq k} |\Gamma^\delta S_{1,1}[\Xi](t, x)| \lesssim \mathcal{B}_{2+\mu, 2+k}[\psi_2 \vec{u}_0] \lesssim \mathcal{A}_{2+\mu, 2+k}[\vec{u}_0] \quad (7.31)$$

for any $(t, x) \in [0, T) \times \overline{\Omega}$.

Now we proceed estimating $S_3[\Xi]$. Because $(1 - \psi_2)\Xi \in X_3(T)$ for any $\Xi \in X(T)$, taking $\rho = 1 - \mu$ in (7.21) we get

$$\begin{aligned} \langle t \rangle^{1/2} \sum_{|\delta| \leq k} |\Gamma^\delta S_3[\Xi](t, x)| &\lesssim \\ &\lesssim \langle t \rangle^{-1/2+\mu} \left(\|u_0\|_{H^{k+2}(\Omega_3)} + \|u_1\|_{H^{k+1}(\Omega_3)} + \log(e+t) \sum_{|\beta| \leq k+1} \|\langle s \rangle^{1-\mu} \partial^\beta f(s, y)\|_{L_t^\infty L^\infty(\Omega_3)} \right) \end{aligned} \quad (7.32)$$

for $(t, x) \in [0, T) \times \overline{\Omega}$.

By using the trivial inequality $\langle s \rangle^{1-\mu} \lesssim |y|^{1/2} W_{1,1}(s, y)$ in $[0, T) \times \Omega_3$, from (7.30), (7.31) and (7.32) we can conclude that

$$\begin{aligned} \langle t \rangle^{1/2} \sum_{|\delta| \leq k} |\Gamma^\delta S_1[\Xi]| + \langle t \rangle^{1/2} \sum_{|\delta| \leq k} |\Gamma^\delta S_3[\Xi]| &\lesssim \\ &\lesssim \langle t \rangle^{-(1/2)+\mu} \mathcal{A}_{2+\mu, 2+k}[\vec{u}_0] + \log(e+t) \sum_{|\beta| \leq 2+k} \| |y|^{1/2} W_{1,1+\mu}(s, y) \Gamma^\beta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned} \quad (7.33)$$

Finally we consider the terms $S_2[\Xi]$, $S_4[\Xi]$. Let us set $g_j[\Xi] = (\partial_t^2 - \Delta)S_j[\Xi]$ for $j = 2, 4$. Recalling the definition of L_0 , we find

$$\begin{aligned} g_2[\Xi] &= -[\psi_2, -\Delta]L[[\psi_1, -\Delta]S_0[\psi_2 \Xi]]; \\ g_4[\Xi] &= -[\psi_3, -\Delta]S[(1 - \psi_2)\Xi]. \end{aligned}$$

Having in mind (7.27) we can say that g_2 and g_4 have the same structures as S_1 and S_3 , but they contain one more derivative. Therefore, arguing similarly to the derivation of (7.33), we arrive at

$$\begin{aligned} \langle t \rangle^{1/2} \sum_{|\delta| \leq k} |\Gamma^\delta g_2[\Xi]| + \langle t \rangle^{1/2} \sum_{|\delta| \leq k} |\Gamma^\delta g_4[\Xi]| &\lesssim \\ &\lesssim \langle t \rangle^{-(1/2)+\mu} \mathcal{A}_{2+\mu, 3+k}[\vec{u}_0] + \log(e+t) \sum_{|\beta| \leq 3+k} \| |y|^{1/2} W_{1,1+\mu}(s, y) \Gamma^\beta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned} \quad (7.34)$$

On the other hand, we have $S_i[\Xi] = L_0[g_i]$ for $i = 2, 4$. Thus, since g_2 and g_4 are supported on $\overline{B_4 \setminus B_2}$, we are in a position to apply (7.22) and we get

$$\begin{aligned} \sum_{|\delta| \leq k} \left(|\Gamma^\delta S_2[\Xi]| + |\Gamma^\delta S_4[\Xi]| \right) (t, x) &\lesssim \\ &\lesssim \mathcal{A}_{2+\mu, 3+k}[\vec{u}_0] + \log(e+t) \sum_{|\beta| \leq 3+k} \| |y|^{1/2} W_{1,1+\mu}(s, y) \Gamma^\beta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned} \quad (7.35)$$

Now (2.8) follows from (7.29), (7.33) and (7.35).

Next we prove (2.10). Trivially one has

$$\begin{aligned} \sum_{|\delta| \leq k} |\Gamma^\delta \partial(\psi_1(x) S_0[\psi_2 \Xi])(t, x)| &\lesssim \\ &\lesssim \sum_{|\delta| \leq k} |\Gamma^\delta \partial S_0[\psi_2 \Xi](t, x)| + \sum_{|\delta| \leq k} |\Gamma^\delta \nabla_x \psi_1(x)| |\Gamma^\delta S_0[\psi_2 \Xi](t, x)|. \end{aligned}$$

Since in Ω one has $|y| \simeq \langle y \rangle$, by (7.9) and (7.13) with $\eta = 1/2$, we see that

$$\begin{aligned} \sum_{|\delta| \leq k} |\Gamma^\delta \partial S_0[\psi_2 \Xi](t, x)| &\lesssim \langle t + |x| \rangle^{-1/2} \langle t - |x| \rangle^{-1/2} \mathcal{A}_{2+\mu, k+1}[\vec{u}_0] + \\ &+ w_{1/2}(t, x) \log(e + t + |x|) \sum_{|\delta| \leq k+1} \| |y|^{1/2} W_{1,1}(s, y) \Gamma^\delta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned}$$

On the other hand, by (7.8) and (7.11) with $\kappa = 1$, we have

$$\begin{aligned} \langle t + |x| \rangle^{1/2} \log^{-1} \left(e + \frac{\langle t + |x| \rangle}{\langle t - |x| \rangle} \right) \sum_{|\delta| \leq k} |\Gamma^\delta S_0[\psi_2 \Xi](t, x)| &\lesssim \\ &\lesssim \mathcal{A}_{3/2, k}[\vec{u}_0] + \log(e + t + |x|) \sum_{|\delta| \leq k} \| |y|^{1/2} W_{1,1}(s, y) \Gamma^\delta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned}$$

Since the logarithmic term on the left-hand side does not appear when $x \in \Omega_2$, we get

$$\begin{aligned} w_{1/2}^{-1}(t, x) \sum_{|\delta| \leq k} \left| \Gamma^\delta \partial(\psi_1(x) S_0[\psi_2 \Xi])(t, x) \right| &\lesssim \\ &\lesssim \mathcal{A}_{2+\mu, k+1}[\vec{u}_0] + \log(e + t + |x|) \sum_{|\delta| \leq k+1} \| |y|^{1/2} W_{1,1}(s, y) \Gamma^\delta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned} \quad (7.36)$$

Therefore, $\partial(\psi_1 S_0[\psi_2 \Xi])$ has the desired bound.

Let us recall that $|x|$ is bounded in $\text{supp } S_1[\Xi](t, \cdot) \cup \text{supp } S_3[\Xi](t, \cdot)$. In particular we get $w_{1/2}^{-1}(t, x) \lesssim \langle t \rangle^{1/2}$. From (7.33) we deduce

$$\begin{aligned} \sum_{|\delta| \leq k} w_{1/2}^{-1}(t, x) \left(|\Gamma^\delta \partial S_1[\Xi](t, x)| + |\Gamma^\delta \partial S_3[\Xi](t, x)| \right) &\lesssim \\ &\lesssim \mathcal{A}_{2+\mu, 3+k}[\vec{u}_0] + \log(e+t) \sum_{|\beta| \leq 3+k} \| |y|^{1/2} W_{1,1+\mu}(s, y) \Gamma^\beta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned} \quad (7.37)$$

As for $S_4[\Xi]$, we use a similar estimate to (7.32) with k replaced by $k+1$, that is

$$\begin{aligned} \langle t \rangle^{1-\mu} \sum_{|\delta| \leq k+1} |\Gamma^\delta g_4[\Xi](t, x)| &\lesssim \\ &\lesssim \mathcal{A}_{2+\mu, k+4}[\vec{u}_0] + \log(e+t) \sum_{|\beta| \leq k+3} \| |y|^{1/2} W_{1,1+\mu}(s, y) \Gamma^\beta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned} \quad (7.38)$$

Applying (7.23) with $\rho = 1 - \mu$ and $\eta = \mu$ ($0 < \mu \leq 1/4$), we find that

$$\begin{aligned} \sum_{|\delta| \leq k} w_{1-2\mu}^{-1}(t, x) |\Gamma^\delta \partial S_4[\Xi](t, x)| &\lesssim \\ &\lesssim \mathcal{A}_{2+\mu, k+4}[\vec{u}_0] + \log(e+t) \sum_{|\beta| \leq k+3} \| |y|^{1/2} W_{1,1+\mu}(s, y) \Gamma^\beta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned} \quad (7.39)$$

For treating $S_2[\Xi]$, we decompose $g_2[\Xi]$ into $g_{2,1}[\Xi]$ and $g_{2,2}[\Xi]$ as was done for evaluating $S_1[\Xi]$. Then $L_0[g_{2,1}]$ can be estimated as $S_4[\Xi]$. On the other hand, using (7.23) with $\rho = 1/2$ and $\eta = 0$ for $L_0[g_{2,2}]$, we arrive at

$$\begin{aligned} \sum_{|\delta| \leq k} w_{1/2}^{-1}(t, x) |\Gamma^\delta \partial S_2[\Xi](t, x)| &\lesssim \\ &\lesssim \mathcal{A}_{2+\mu, 4+k}[\vec{u}_0] + \log^2(e+t+|x|) \sum_{|\beta| \leq 4+k} \| |y|^{1/2} W_{1,1+\mu}(s, y) \Gamma^\beta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned} \quad (7.40)$$

Thus we obtain (2.10) from (7.36), (7.37), (7.39), and (7.40).

In order to show (2.9), we remark that $w_{1/2} \leq w_{(1/2)-\eta}$ so that in (7.36) we can replace $w_{1/2}$ with $w_{1/2-\eta}$. Moreover, (7.37) and (7.38) hold with $\mu = 0$ if we replace $\log(e+t)$ by $\log^2(e+t)$, thanks to (7.24) with $\kappa = 1$. Therefore, the application of (7.23) with $\rho = 1/2$ and $0 < \eta < 1/2$ leads to (7.39) with $w_{1/2}^{-1}$ replaced by $w_{(1/2)-\eta}^{-1}$ and $\mu = 0$ in the second term of the right-hand side. Hence we get (2.9).

Finally, we prove (2.11). We put $\eta' = \eta/2$. By (7.9) and (7.13), we see that

$$\begin{aligned} \sum_{|\delta| \leq k+1} |\Gamma^\delta \partial_t(\psi_1(x) S_0[\psi_2 \Xi](t, x))| &\lesssim \sum_{|\delta| \leq k+1} |\Gamma^\delta \partial_t S_0[\psi_2 \Xi](t, x)| \lesssim \\ &\lesssim \langle t+|x| \rangle^{-1/2} \langle t-|x| \rangle^{-1/2} \mathcal{A}_{2+\mu, k+2}[\vec{u}_0] + \\ &\quad + w_{1-\eta'}(t, x) \log(e+t+|x|) \sum_{|\delta| \leq k+2} \| |y|^{1/2} W_{1,1}(s, y) \Gamma^\delta f(s, y) \|_{L_t^\infty L_\Omega^\infty}. \end{aligned}$$

Therefore, $\partial_t(\psi_1 S_0[\psi_2 \Xi])$ has the desired bound because $w_{1-\eta'} \leq w_{1-\eta}$.

Combining this estimate with (7.21), we obtain the estimate for $S_1[\Xi]$. Indeed, for $0 < \eta < 1$ we have

$$\langle t \rangle^{1-\eta'} \sum_{|\delta| \leq k+1} |\Gamma^\delta \partial_t S_1[\Xi](t, x)| \lesssim \log(e+t) \sum_{|\beta| \leq k+2} \| \langle s \rangle^{1-\eta'} \partial^\beta \partial_t([\psi_1, -\Delta] S_0[\psi_2 \Xi])(s, y) \|_{L_t^\infty L^\infty(\Omega_2)}.$$

Recalling (7.27), we can use the estimate of $\partial_t(\psi_1 S_0[\psi_2 \Xi])$ adding two derivatives. In conclusion, we have

$$\langle t \rangle^{1-\eta'} \sum_{|\delta| \leq k+1} |\Gamma^\delta \partial_t S_1[\Xi](t, x)| \lesssim \Theta_{\mu, k+4}(t)$$

for $(t, x) \in [0, T) \times \overline{\Omega}$, where

$$\Theta_{\mu, m}(t) := \mathcal{A}_{2+\mu, m}[\vec{u}_0] + \log^2(e+t) \sum_{|\delta| \leq m} \| |y|^{1/2} W_{1,1}(s, y) \Gamma^\delta f(s, y) \|_{L_t^\infty L_\Omega^\infty}.$$

Since we have $(1 - \psi_2)\Xi \in X_3(T)$ for any $\Xi \in X(T)$, by using (7.21) with $\rho = 1 - \eta'$ we have

$$\langle t \rangle^{1-\eta'} \sum_{|\delta| \leq k+1} |\Gamma^\delta \partial_t S_3[\Xi](t, x)| \lesssim \Theta_{\mu, k+3}(t).$$

In order to treat $S_2[\Xi]$ and $S_4[\Xi]$, we set $g_j[\Xi] = (\partial_t^2 - \Delta)S_j[\Xi]$ for $j = 2, 4$ as before. Going similar lines to the estimates for $S_1[\Xi]$ and $S_3[\Xi]$, with a derivative more, we can reach at

$$\langle t \rangle^{1-\eta'} \sum_{|\delta| \leq k+1} |\Gamma^\delta \partial_t g_2[\Xi]| + \langle t \rangle^{1-\eta'} \sum_{|\delta| \leq k+1} |\Gamma^\delta \partial_t g_4[\Xi]| \lesssim \Theta_{\mu, k+5}(t).$$

Let us recall that g_2 and g_4 are supported on $\overline{B_4} \setminus B_2$ and $\partial_t S_i[\Xi] = L_0[\partial_t g_i]$ for $i = 2, 4$. We are in a position to apply (7.23) (with $\rho = 1 - \eta'$, and η replaced by η') and obtain

$$w_{1-\eta}^{-1}(t, x) \sum_{|\delta| \leq k} \sum_{i=2,4} |\Gamma^\delta \partial_t S_i[\Xi](t, x)| \lesssim \sum_{i=2,4} \sum_{|\delta| \leq k+1} \| \langle s \rangle^{1-\eta'} \partial^\beta \partial_t g_i(s, y) \|_{L_t^\infty L^\infty(\Omega_4)} \lesssim \Theta_{\mu, k+5}(t).$$

The proof of Theorem 2.1 is complete. \square

Remark 7.1. The main difference between the Dirichlet and the Neumann boundary cases is in the logarithmic loss in the local energy decay estimate (7.16). Due to this term, comparing our result with the one in [K12], we see that the estimates for $S_2[\Xi]$ and $S_4[\Xi]$ are worse in the Neumann case.

APPENDIX: A LOCAL EXISTENCE THEOREM OF SMOOTH SOLUTIONS

Here we sketch a proof of the following local existence theorem for the semilinear case (for the general case, see [SN89]). We underline that the convexity assumption for the obstacle is not necessary for the local existence result.

Theorem A.1. *Let \mathcal{O} be a bounded obstacle with C^∞ boundary and $\Omega = \mathbb{R}^2 \setminus \mathcal{O}$. For any $\phi, \psi \in C_0^\infty(\overline{\Omega})$ satisfying the compatibility condition of infinite order and*

$$\|\phi\|_{H^5(\Omega)} + \|\psi\|_{H^4(\Omega)} \leq R, \tag{A.1}$$

there exists a positive constant $T = T(R)$ such that the mixed problem (1.1) admits a unique solution $u \in C^\infty([0, T) \times \overline{\Omega})$. Here T is a constant depending only on R .

For nonnegative integer s , we put

$$Y_T^s := \bigcap_{j=0}^s C^j([0, T]; H^{s-j}(\Omega)),$$

and

$$\|h\|_{Y_T^s} := \sum_{j=0}^s \sup_{t \in [0, T]} \|\partial_t^j h(t, \cdot)\|_{H^{s-j}(\Omega)}.$$

Let v_j for $j \geq 0$ be given as in Definition 1.1. First we show the following result.

Lemma A.1. *Let $m \geq 2$. Suppose that $(\phi, \psi) \in H^{m+2}(\Omega) \times H^{m+1}(\Omega)$ satisfies the compatibility condition of order $m+1$, that is to say, $\partial_\nu v_j|_{\partial\Omega} = 0$ for $j \in \{0, 1, \dots, m+1\}$, and*

$$\|\phi\|_{H^{m+2}(\Omega)} + \|\psi\|_{H^{m+1}(\Omega)} \leq M. \quad (\text{A.2})$$

Then¹, there exists a positive constant $T = T(m, M)$ such that the mixed problem (1.1) admits a unique solution $u \in Y_T^{m+2}$. Here T is a constant depending only on m and M .

Proof. To begin with, we note that the Sobolev embedding theorem implies

$$\sum_{|\beta| \leq [(m+1)/2]+1} \|\partial^\beta h(t, \cdot)\|_{L^\infty_\Omega} \lesssim \sum_{|\beta| \leq [(m+1)/2]+3} \|\partial^\beta h(t, \cdot)\|_{L^2_\Omega} \leq \sum_{|\beta| \leq m+2} \|\partial^\beta h(t, \cdot)\|_{L^2_\Omega} \quad (\text{A.3})$$

for $m \geq 2$.

We show the existence of u by constructing an approximate sequence $\{u^{(n)}\} \subset Y_T^{m+2}$, and proving its convergence for suitably small $T > 0$. Throughout this proof, C_M denotes a positive constant depending on M , but being independent of T . In order to keep the compatibility condition, we need to choose an appropriate function for the first step: for a moment, we suppose that we can choose a function $u^{(0)} \in Y_T^{m+2}$ satisfying $(\partial_t^j u^{(0)})(0, x) = v_j$ for all $j \in \{0, 1, \dots, m+2\}$. For $n \geq 1$ we inductively define $u^{(n)}$ as

$$u^{(n)} = S[\phi, \psi, G(\partial u^{(n-1)})]. \quad (\text{A.4})$$

We have to check that $u^{(n)}$ is well defined. Let $v_0^{(n)} := \phi$, $v_1^{(n)} := \psi$, and $v_j^{(n)} := \Delta v_{j-2}^{(n)} + \partial_t^{j-2}(G(\partial u^{(n-1)}))|_{t=0}$ for $j \geq 2$. Suppose that $u^{(n-1)} \in Y_T^{m+2}$ with $(\partial_t^j u^{(n-1)})(0) = v_j$ for $0 \leq j \leq m+2$. Then we can see that $v_j^{(n)} = v_j$ for $0 \leq j \leq m+2$, and consequently the compatibility condition of order $m+1$ is satisfied for the equation of $u^{(n)}$. Since (A.3) implies $G(\partial u^{(n-1)}) \in Y_T^{m+1}$, the linear theory (see [I68]) shows that $u^{(n)} \in Y_T^{m+2}$. Therefore, by induction with respect to n , we see that $\{u^{(n)}\} \subset Y_T^{m+2}$ is well defined, and that $(\partial_t^j u^{(n)})(0) = v_j^{(n)} = v_j$ for $0 \leq j \leq m+2$ and $n \geq 0$.

Now we are going to explain how to construct $u^{(0)}$. We can show that $v_j \in H^{m+2-j}(\Omega)$ for $0 \leq j \leq m+2$ by its definition and (A.3). By the well-known extension theorem, there is $V_j \in H^{m+2-j}(\mathbb{R}^2)$ such that $V_j|_\Omega = v_j$ and $\|V_j\|_{H^{m+2-j}(\mathbb{R}^2)} \lesssim \|v_j\|_{H^{m+2-j}(\Omega)}$. Let $(a_{kl})_{0 \leq k, l \leq m+2}$ be the inverse matrix of $(i^k(l+1)^k)_{0 \leq k, l \leq m+2}$, where $i = \sqrt{-1}$. We put

$$\widehat{V}(t, \xi) = \sum_{k, l=0}^{m+2} \exp(i(k+1)\langle \xi \rangle t) a_{kl} \widehat{V}_l(\xi) \langle \xi \rangle^{-l},$$

where \widehat{V}_l is the Fourier transform of V_l . We set $u^{(0)}(t) = V(t)|_\Omega$ with the inverse Fourier transform $V(t)$ of $\widehat{V}(t)$. Now we can show that $u^{(0)}(t)$ has the desired property, and $\|u^{(0)}\|_{Y_T^{m+2}} \leq C_M$ (see [SN89] where this kind of function is used to reduce the problem to the case of zero-data).

¹The assumption on initial data here is just for simplicity, and we can prove the same result for initial data with compatibility condition of order m in fact.

Now we are in a position to show that $u^{(n)}$ converges to a local solution of (1.1) on $[0, T]$ with appropriately chosen T . For simplicity of description, we put

$$\|h(t)\|_k = \sum_{j=0}^{m+2-k} \|\partial_t^j h(t)\|_{H^k(\Omega)}$$

for $0 \leq k \leq m+2$. Note that we have $\|h\|_{Y_T^{m+2}} \lesssim \sup_{t \in [0, T]} \sum_{k=0}^{m+2} \|h(t)\|_k$. We also set $G_n(t, x) = G(\partial u^{(n)}(t, x))$ for $n \geq 0$. Combining the elementary inequality

$$\|h(t)\|_{L_\Omega^2} \leq \|h(0)\|_{L_\Omega^2} + \int_0^t \|(\partial_t h)(\tau)\|_{L_\Omega^2} d\tau$$

with the standard energy inequality for $\partial_t^j u^{(n)}$ with $0 \leq j \leq m+1$, we get

$$\|u^{(n)}(t)\|_0 + \|u^{(n)}(t)\|_1 \leq (1+T) \left(C_M + C \sum_{j=0}^{m+1} \int_0^t \|(\partial_t^j G_{n-1})(\tau)\|_{L_\Omega^2} d\tau \right).$$

Writing

$$\Delta \partial^\beta u^{(n)}(t, x) = \partial_t^2 \partial^\beta u^{(n)} - (\partial^\beta G_{n-1})(0, x) - \int_0^t (\partial_t \partial^\beta G_{n-1})(\tau, x) d\tau$$

for a multi-index β and using the elliptic estimate, given in Lemma 2.2, we have

$$\|u^{(n)}(t)\|_k \leq C \left(\|u^{(n)}(t)\|_{k-2} + \|u^{(n)}(t)\|_{k-1} + C_M + \sum_{|\alpha| \leq k-1} \int_0^t \|(\partial^\alpha G_{n-1})(\tau)\|_{L_\Omega^2} d\tau \right)$$

for $2 \leq k \leq m+2$. By induction we get control of $\|u^{(n)}(t)\|_k$ for $0 \leq k \leq m+2$, and obtain

$$\sum_{k=0}^{m+2} \|u^{(n)}(t)\|_k \leq (1+T) \left(C_M + C \sum_{|\alpha| \leq m+1} \int_0^t \|(\partial^\alpha G_{n-1})(\tau)\|_{L_\Omega^2} d\tau \right). \quad (\text{A.5})$$

It follows from (A.3) that

$$\sum_{|\alpha| \leq m+1} \|(\partial^\alpha G_{n-1})(\tau)\|_{L_\Omega^2} \leq C \|u^{(n-1)}\|_{Y_T^{m+2}}^3, \quad 0 \leq \tau \leq T, \quad (\text{A.6})$$

and (A.5) implies $\|u^{(n)}\|_{Y_T^{m+2}} \leq (1+T) \left(C_M + CT \|u^{(n-1)}\|_{Y_T^{m+2}}^3 \right)$ for $n \geq 1$. From this, if we take appropriate constants N_M and T_M which can be determined by M , we can show that $\|u^{(n)}\|_{Y_T^{m+2}} \leq N_M$ for all $n \geq 0$, provided that $T \leq T_M$. In the same manner, we can also show that there is some $T'_M (\leq T_M)$ such that

$$\|u^{(n+1)} - u^{(n)}\|_{Y_T^{m+2}} \leq \frac{1}{2} \|u^{(n)} - u^{(n-1)}\|_{Y_T^{m+2}}$$

for all $n \geq 1$, provided that $T \leq T'_M$. Now we see that if $T \leq T'_M$, then $\{u^{(n)}\}$ is a Cauchy sequence in Y_T^{m+2} , and there is $u \in Y_T^{m+2}$ such that $\lim_{n \rightarrow \infty} \|u^{(n)} - u\|_{Y_T^{m+2}} = 0$. It is not difficult to see that this u is the desired solution to (1.1).

Uniqueness can be easily obtained by the energy inequality. \square

Theorem A.1 is a corollary of Lemma A.1.

Proof of Theorem A.1. The assumption on the initial data guarantees that for each $m \geq 3$, there is a positive constant M_m such that $\|\phi\|_{H^{m+2}(\Omega)} + \|\psi\|_{H^{m+1}(\Omega)} \leq M_m$. Hence, by Lemma A.1, there is $T_m = T(m, M_m) > 0$ such that (1.1) admits a unique solution $u \in Y_{T_m}^{m+2}$. Note that we may take $T_3 = T(3, R)$. We put

$$C_0 := \|u\|_{Y_{T_3}^{3+2}}. \quad (\text{A.7})$$

Our aim is to prove that (1.1) admits a solution $u \in \bigcap_{m \geq 3} Y_{T_3}^{m+2}$. Then the Sobolev embedding theorem implies that $u \in C^\infty([0, T_3] \times \overline{\Omega})$, which is the desired result. For this purpose, we are going to prove the following *a priori* estimate: for each $m \geq 3$, if $u \in Y_T^{m+2}$ is a solution to (1.1) with some $T \in (0, T_3]$, then there is a positive constant C_m , which is independent of T , such that

$$\|u(t)\|_{Y_T^{m+2}} \leq C_m. \quad (\text{A.8})$$

Once we obtain this estimate, by applying Lemma A.1 repeatedly, we can see that $u \in Y_{T_3}^{m+2}$ for each $m \geq 3$, which concludes the proof of Theorem A.1.

Now we show (A.8) by induction. For $m = 3$ (A.8) follows immediately from (A.7). Suppose that we have (A.8) for some $m = l \geq 3$. If we put

$$\|h(t)\|_k = \sum_{j=0}^{l+3-k} \|\partial_t^j h(t)\|_{H^k(\Omega)},$$

then, similarly to (A.5), we obtain

$$\sum_{k=0}^{l+3} \|u(t)\|_k \leq (1 + T_3) \left(C + C \sum_{|\alpha| \leq l+2} \int_0^t \|(\partial^\alpha (G(\partial u(\tau))))\|_{L_\Omega^2} d\tau \right).$$

Since $[(m+1)/2] + 3 \leq m+1$ for $m \geq 4$, we have

$$\sum_{|\beta| \leq [(m+1)/2] + 1} \|\partial^\beta h(t, \cdot)\|_{L_\Omega^\infty} \leq C \sum_{|\beta| \leq m+1} \|\partial^\beta h(t, \cdot)\|_{L_\Omega^2}, \quad m \geq 4, \quad (\text{A.9})$$

in place of (A.3). Combining this estimate for $m = l+1$ with the inductive assumption, we get

$$\sum_{|\alpha| \leq l+2} \|\partial^\alpha (G(\partial u(\tau)))\|_{L_\Omega^2} \leq C C_l^2 \sum_{k=0}^{l+3} \|u(\tau)\|_k,$$

which yields

$$\sum_{k=0}^{l+3} \|u(t)\|_k \leq (1 + T_3) \left(C + C C_l^2 \int_0^t \sum_{k=0}^{l+3} \|u(\tau)\|_k d\tau \right).$$

Now the Gronwall Lemma implies $\sum_{k=0}^{l+3} \|u(t)\|_k \leq C(1 + T_3) \exp(C C_l^2 (1 + T_3) T_3) =: C_{l+1}$ for $0 \leq t \leq T(\leq T_3)$, which implies $\|u\|_{Y_T^{l+3}} \leq C_{l+1}$ for $0 \leq T \leq T_3$. This completes the proof of (A.8). \square

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